AN ADVANCED COURSE IN ALGEBRA

REVISED EDITION

MILLER and ROURKE

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AN ADVANCED COURSE IN ALGEBRA

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BY

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AND

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REVISED EDITION

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PREFACE

SINCE the beginning of the century, increasing attention has been given by educators to the content and the teaching of algebra. This has been partly forced by the trend, in some countries, toward diminishing the mathematics requirement in the schools. Quite independently of this trend, however, many prominent educators have given direction to the reform movement in the teaching of mathematics. This direction is indicated by the following quotations:

> "Too much time has been spent in the past on the tricks of simplifying expressions and solving equations."
>
> (C. W. Spriggs, in *School Certificate Algebra*)
>
> "One of the causes of the apparent triviality of much of elementary algebra is the preoccupation of the text-books with the solution of equations."
>
> (A. N. Whitehead, in *Introduction to Mathematics*)
>
> "Functional mathematics is neither a technical skill nor a formal discipline but a mode of thinking."
>
> (H. R. Hamley, in *Functional Thinking*)
>
> "There can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other."
>
> (Report of the National Committee on the Reorganization of Mathematics in Secondary Schools)

The point of view expressed in this last statement is amplified in the following quotation from one of the foremost exponents of the reform movement in America.

> "Algebra emerges strengthened and beautified, no longer needing an apologist, but manifesting itself as a true need of the modern world which is, both in its manifold scientific enterprises and in its everyday affairs, vitally interested in controlling and interpreting the relations between varying quantities."
>
> (E. R. Hedrick, in *School Science and Mathematics*)

These quotations indicate the general agreement that emphasis should be placed, not on the manipulation of expressions and equations but rather on the use and the interpretation of relationships between variable quantities. This idea is further developed in the *Introduction* preceding Chapter I.

In accordance with these views, the present text aims to present the various topics of a second course in school algebra in the light of the function concept. Consequently the relational aspects of the graph, the table of values, the formula, and the equation are emphasized throughout the book. As a result of his first course in algebra, the pupil should already have a background of experience with situations in which one variable depends on another. Building on this experience, the present text begins with the definition of a function and proceeds to coordinate and extend the pupil's knowledge of special types of functions. This order need not necessarily be followed by a teacher; he may prefer to begin with a subsequent chapter such as that on the progressions, returning later to the general considerations regarding functions.

A word of explanation is in order regarding the problems, of which a large number is provided in the text. While the need for drill problems has not been overlooked, the main object in the selection and arrangement of exercises has been to encourage the formulation of data in precise terms and the interpretation of results. Hence long lists of almost identical exercises are avoided. After the introduction of a new topic, illustrative examples are worked in the text and, following these, a selection of exercises is included, to give the pupil the opportunity of organizing his knowledge of this topic. At the end of each chapter and within some of the chapters, larger selections of graded exercises are inserted, covering a wider range of ideas. It is by working such sets of exercises that a student gains mastery of the subject. The exercises graded "A" are simple and direct in character and should be worked by all pupils; those graded "B" make somewhat greater demands but are not beyond the powers of the average pupil; the "C" exercises provide opportunities for pupils with special ability in mathematics. At the end of the book there is added a group of 100 exercises, diversified in character.

Chapter X is intended primarily as an application of the geometric series to the problems of finance. However, attention has also been given to the business aspects of the problems, in order to furnish a suitable introduction to the mathematics of finance.

For most of the problems which admit brief answers, these are given. The policy adopted was to furnish the answer when, by so doing, the usefulness of the problem was not impaired. For the problems of Chapter X, all answers are given correct to five figures.

In the writing of the book, the authors have received inspiration from many sources. In particular, they desire to pay tribute to the *Report of the National Committee for the Reorganization of Mathematics in Secondary Schools,* H. R. Hamley's treatise, *Functional Thinking,* and T. P. Nunn's *Exercises in Algebra.* Some problems have been adapted from the examination papers of the Ontario Department of Education. The authors have profited by suggestions from Professor I. R. Pounder, Mr. C. W. Robb, and Mr. W. A. Jackson. These suggestions are gratefully acknowledged. The authors would welcome further suggestions or criticisms from teachers who use the book.

N. M.
R. E. K. R.

PREFACE TO THE REVISED EDITION

THE revised edition differs from the original chiefly in the provision of a larger number of exercises, most of which are simple and direct in character and designed to furnish further drill on the fundamentals. Parts of the text have been rewritten or amplified, in places where experience has indicated simpler or more complete explanations.

In response to requests from a number of teachers, sets of review exercises on the basic skills are included at the back of the book. The authors believe that the soundest approach to algebra at this level is through the medium of new ideas rather than through the threshing of old manipulative straw. However, if the skill in algebraic operations is not adequate for the appreciation of new concepts, then it is hoped that the review sets will provide opportunities for sufficient improvement.

The book contains some supplementary sections which are not necessary to the continuity of the text. These sections, marked with a star, may be omitted at a first reading.

The authors are grateful to many teachers who have expressed satisfaction in using the book and whose constructive suggestions have largely formed the basis for this revision.

N. M.
R. E. K. R.

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INTRODUCTION

The adage concerning the trees obscuring the forest has point in many situations and in none more than in the study of a branch of mathematics. Therefore, before the student becomes immersed in the details of the subject set forth in this book, it may be well to glance at its broad outlines and to point out some of the objectives in its study. The student who is beginning this course has already some familiarity with algebraic methods and has discovered the power of algebra as a tool for solving problems. This power results from the use of letters for numbers and of symbols for operations ($+$, $-$, $\times$, $\div$, etc.). Such notation makes it possible to express, in condensed form, many relationships between two magnitudes which change so that one depends on the other. Thus $2x^2-5x+4$ depends for its value on x, and this expression gives the relationship in a much more terse and usable fashion than an equivalent description in words.

Relationships between variables occur constantly in our experience. In some of these, the relationship can be simply expressed; in others, it is not of a simple character. As examples, consider the relations between the time taken on a journey and the speed of travel, the force needed to drive a car up hill and the steepness of the hill, the amount of gas used in a stove and the time taken for cooking, the parcel post rate and the weight of the parcel. Other examples will occur in the following chapters. Men and women, in their daily affairs, have continually to take account of these and other relationships—in instalment buying, in purchasing insurance, in paying income taxes, in using recipes, in mixing feeds, cement or fertilizers, and in many other situations.

Examples such as these, however, do not exhaust the importance of the idea of relationship or dependence of variables. It permeates all our thinking. How often, in conversation and discussion, we hear the phrases "It all depends", "It depends on this or that factor"! Many of the differences which separate men on political, social or other questions arise from a lack of knowledge concerning the factors which affect their conclusions. This is, in many cases, unavoidable but it serves to emphasize the value of exploring one small corner of knowledge which gives practice in stating with exactness on what things certain others depend and the precise nature of this dependence. Our understanding of many subjects other than mathematics will grow according to the precision with which we are able to estimate the number and character of the dependence relations.

From the beginning of a first course, algebra makes use of variables of which, in a given problem, one depends on others. As the subject develops in the present course, this relational or functional aspect is made more explicit and forms a unifying thread which runs through all the various topics considered. One of the most useful ways of illustrating and studying relationships is by the use of graphs. Graphs are to the student of algebra what plans are to the architect—they illuminate and vivify the subject. They are plentifully used in our subsequent work.

A considerable part of elementary algebra has as its purpose the solution of problems which have to do with arithmetic, commerce, physics, or other subjects. It adds to our satisfaction in studying a subject to be able to make use of it in so-called practical ways. There is no lack of such applications in the present course—the use of the geometric series in solving problems of finance is an illustration.

On the other hand, the student has now reached the point where it is proper to call his attention to the fact that algebra, like any other branch of mathematics, is its own justification. The real answer to the question, "Why study algebra?" is the same as to the questions, "Why read poetry?", "Why listen to a Beethoven symphony?" The man who, when asked why he spent his spare time working at problems, replied that he did it for fun, gave a sufficient answer; he might have expressed his answer in more sophisticated language—that he found, in the satisfaction of his curiosity, his real happiness. A critic might object that working at such a subject as algebra without a definite practical end in view—such as the building of bridges or the calculation of insurance premiums—might perhaps be good fun but would not lead to results useful to the scientist or the engineer. The verdict of history is just the opposite; on many occasions, mathematical theories, which were developed without any thought of applications, have proved, long after, to be the necessary tools for scientific problems.

The student will find it helpful to keep in mind the two aspects of the subject—its usefulness and its interest—as he reads the following pages. He will find in them an abundance of exercises with which to familiarize himself with the subject and to develop his skill; he will find various points at which the subject makes contact with his experience in other lines; finally, it is hoped that his curiosity will grow with his knowledge and that he will find the subject a source of pleasure and satisfaction.

AN ADVANCED COURSE IN ALGEBRA

CHAPTER I

FUNCTIONS

1. Real numbers. The subject of algebra grows out of arithmetic and, like arithmetic, it deals with numbers. In a study of the subject, it is necessary from time to time to re-examine the system of numbers on which the structure of algebra is built. Our first examination aims to make clear the point of view of algebra towards numbers, as distinct from that of arithmetic.

Numbers present themselves for study in problems either singly or in groups of more than one. A group of numbers which is specified in some definite manner will be called, for convenience, a *set* of numbers. The following are examples:

(*a*) the set of all positive integers, 1, 2, 3, $\cdots$;
(*b*) the set of positive integers from 1 to 100, inclusive ;
(*c*) the set of all positive numbers less than 10 ;
(*d*) the set of all numbers obtained from x^2 by giving to x integral values.

A graphical method of representing numbers is obtained by using a straight line after the manner of one of the axes employed in the plotting of graphs. If a point on this line is chosen as origin and a distance of 5 units is measured off in a chosen direction, called positive, the point which is arrived at will represent the number 5. If a distance of 3 units is measured off from the origin in the opposite direction, the point arrived at will represent —3. It will be assumed here as an axiom that every point on the line represents a number, positive, negative, or zero.

The totality of numbers which are represented by all points on the line will be called ***the set of all real numbers.***

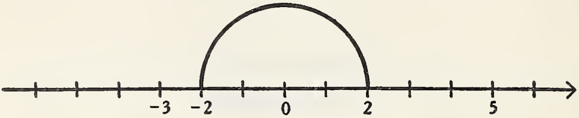


FIG. 1

2. Constants and variables. The student is already aware of the greater power of algebra as compared with arithmetic in the solution of problems. This power results from the use of letters to represent numbers. A letter may be used to stand for a definite number, or it may be used to stand for any one of a set of numbers.

DEFINITION. *A symbol in a given problem is called a* ***constant*** *if it represents the same number throughout the problem.*

Some symbols, such as 2, -3, $\frac{1}{2}$, represent the same constants in all problems. Another example of the same sort is the symbol π, which represents the constant ratio of the circumference of any circle to its diameter.

A letter, such as a, n, k, may be a constant in a given problem without having any specific numerical value attached to it. If the expression or the problem in which the constant occurs has a meaning for any one of a set of values assigned to the constant, the latter is called an ***arbitrary constant.*** Suppose, for example, that, in the expression $2x+b$, b remains constant while x varies. By giving to b different constant values, we get distinct expressions represented by $2x+b$. Then b, in this expression, is an arbitrary constant. *It is arbitrary in the sense that any value may be chosen for it; it is constant in the sense that, when a value has been chosen, it must be held fast throughout the problem in which the expression is used.* Similarly, in $ax+5$, a may be an arbitrary constant and, in $ax+b$, both a and b may be arbitrary constants.

DEFINITION. *A symbol, such as x, y, t, u, is called a* ***variable*** *if it represents any one of a set of numbers.*

Thus, when a variable occurs in a problem, attention is focused on the fact that it is subject to change. The set of values which a variable can take is called its ***range of values.*** Some common ranges of values of a variable x are defined as follows:

(*a*) the positive integers, $1, 2, 3, \cdots$;
(*b*) the positive integers from 1 to n inclusive, where n is a known integer;
(*c*) all real numbers;
(*d*) all real numbers except a constant a;
(*e*) all real numbers greater than a; in this case the variable x satisfies the inequality $x>a$;
(*f*) the number a together with all numbers greater than a; in this case, $x \geqq a$;
(*g*) all real numbers between a and b, of which $a<b$; in this case, $a<x<b$;
(*h*) the numbers a and b together with all numbers between them; in this case, $a \leqq x \leqq b$.

In each of the cases (*g*) and (*h*), the range of values may be described geometrically as an ***interval.*** In case (*h*), the interval includes its end points a and b, while in case (*g*), the interval does not include its end points. If x lies within an interval from $-a$ to a (where a is positive), this is sometimes indicated by the inequality $|x|<a$. The symbol $|x|$ denotes the *absolute or numerical value* of x; thus $|-5|=|5|=5$.

By a slight extension of the definitions, the terms constant and variable will be used for physical quantities of which the measures are constant or variable, respectively. Thus, the amount of time in a day and the distance between two fixed points are constants. If a stone falls from a position of rest, the distance it has fallen, the time it has taken and its speed, all measured at a point in its path which moves with the stone, are variables.

EXERCISES

1. Indicate on a line each of the following ranges of values of the variables named:

(*a*) $x>4$, (*b*) $x\leqq-2$, (*c*) $0<x<10$, (*d*) $-3<t\leqq4$, (*e*) $-1\leqq r\leqq1$.

2. Represent by means of inequalities the following ranges of values of the variable x: (*a*) any value greater than 2; (*b*) any value less than or equal to 2; (*c*) -3 to 5, excluding the ends of the interval; (*d*) -3 to 5, including the ends of the interval; (*e*) -6 to 2, inclusive; (*f*) a to b, exclusive of the ends, where a is less than b; (*g*) m to n, including m but not n, where m is less than n.

3. Represent in *two* ways by means of inequalities each of the following ranges of values of x: (*a*) -1 to 1, inclusive; (*b*) -7 to 7, exclusive; (*c*) all values except those between -5 and 5; (*d*) all values except 100, -100 and the numbers between them; (*e*) all values numerically greater than the positive number b.

3. Functions of a variable. Problems in which only one variable occurs are rare and relatively uninteresting. On the other hand, a fertile field for the use of algebra is furnished by problems each involving two variables which are related to each other in a definite manner.* For illustration, we may think of such relations as that between

(*a*) the side and the perimeter of a variable square ;

(*b*) the radius and the area of a variable circle ;

(*c*) the interest on a sum of money and the principal sum, considered as a variable ;

---

*The student is advised to read the discussion of relationship or dependence in the Introduction.

(*d*) a taxpayer's income and his income tax, for a wide range of incomes ;

(*e*) the pressure and the volume of a given mass of gas ;

(*f*) the temperature of the air and the altitude above the earth ;

(*g*) the speed of a falling stone and the distance through which it has fallen ;

(*h*) the rate at which gasoline is consumed by a motor car and the speed of the car.

DEFINITION. *If x and y are so related that, when any value of x in a certain range is chosen, a corresponding value of y is thereby determined, then y is called a function** of x. Other letters may be used in place of x and y.

According to this definition, x is a variable. In general, y is also a variable. It is possible, however, that all values of y corresponding to all values of x in its range are the same; the function is then a constant. Thus, if the temperature, T, remains at 65°F. during several hours of the day, then for this range of the time, t, $T=65$ represents the temperature function.

When y is a function of x, x is called the ***independent*** variable and y the ***dependent*** variable. When a relation, such as $7x+3y=5$, holds between x and y, it is possible to regard either variable as independent and the other as dependent. For, we have $x=5/7-3y/7$ and $y=5/3-7x/3$. In a given problem, it is generally clear which variable it is natural to regard as independent.

3.1. Functional Notation. A function of x may be denoted by the symbol $f(x)$ which is read: *f of x.* Other symbols, such as

---

*The word function is commonly used in such statements as "the yield of grain is a function of the rainfall", or "a man's temper is a function of his digestion". This usage conforms to our definition, the implication being that, if the amount of rainfall in a given year were stated, then the yield of grain could be estimated, or that, if the state of a man's digestion at any time were known, his temper could be predicted.

$F(y)$, $g(t)$, $R(z)$, are also used to denote functions, the independent variable in each case being given in parentheses.

If the value $x = 2$, for example, is included in the range of values of x, then *$f(2)$ means the value which $f(x)$ takes when $x=2$.* Thus, $f(2)$ and similarly $f(a)$, where a is fixed, denote constants, not variables.

EXAMPLE 1. If $f(x)=4x-2/x$, find the values of $f(2)$, $f(\frac{1}{2})$, $f(2x)$, $f(x+1)$, and $f(x^2)$.

By substitution, we find

$$f(2)=8-1=7;\quad f(\tfrac{1}{2})=2-4=-2;\quad f(2x)=8x-\frac{1}{x};$$

$$f(x+1)=4x+4-\frac{2}{x+1};\quad f(x^2)=4x^2-\frac{2}{x^2}.$$

EXAMPLE 2. Given that $f(t)=at+b$, $f(4)=10$, and $f(0)=-2$, find a, b, and $f(100)$.

By substituting 4 and 0 for t, we find

$$\begin{aligned} 4a+b &= 10, \\ b &= -2. \end{aligned}$$

Hence $a=3$ and $f(t)=3t-2$. It follows that

$$f(100)=300-2=298.$$

EXERCISES

1. If $f(x)=2x^3-3x^2+3x-2$, find $f(1)$, $f(0)$, $f(-1)$, $f(2)$, and $f(-2)$.

2. If $G(x)=x^2-a^2$, find $G(a)$, $G(-a)$, $G(1/x)$, and $1/G(x)$.

3. If $F(x)=x^3-a^3$, find $F(a)$, $F(-a)$, $F(2a)$, and $F(0)$.

4. If $f(y)=3y-2/y$, find $f(2)$, $f(1/3)$, $f(-y)$, $f(1/y)$, and $1/f(y)$.

5. If $f(z)=z^2+2^z$, find $f(0)$, $f(-1)$, $f(3)$, $f(-z)$, $f(1/z)$, and $1/f(z)$.

6. If $P(x)=x^3-2x^2+2x$, find $P(x+1)$, $P(x-1)$, $P(2x)$, and $P(x^2)$.

7. If $R(x)=\dfrac{x+1}{x-1}$, simplify $R(1/x)+R(-x)-1/R(x)$.

8. If $f(x)=\dfrac{x}{x-1}$, find the value of $f(2x)\cdot f(-2x)-f(4x^2)$.

9. If $f(r)=10(2^{5-r}\cdot 3^r)$, find $f(3)$, $f(5)$, and $f(5-r)$.

10. If $F(x)=2^x$, find $\{F(x)\}^2$, $F(x^2)$, $F\{F(x)\}$, $F\{F(1/x)\}$, and $F\{1/F(x)\}$.

11. If $F(x)=a^x$, show that $F(m)\times F(n)=F(m+n)$. Express in other notation, using the symbol F: $F(m)\div F(n)$, $\{F(m)\}^n$, $\sqrt[n]{F(m)}$.

12. If $F(x)=\dfrac{x^2-4}{x-1}$ and $G(x)=\dfrac{x^2-1}{x+2}$ find the value of $F(-2)+G(1)$, $F(76)\times G(1)$, $F(0)-G(0)$, $F(3)\div G(2)$, $F(a)\times G(a)$.

13. Given that $L(r)=kr+m$, $L(-1)=19$, and $L(1)=5$, find k, m, and $L(-3)$.

14. Given that $P(x)=ax^3+bx$, $P(-2)=8$, and $P(3)=3$, find a, b, and $P(4)$.

4. **Examples of functions.** The definition of a function (page 5) gives no indication of a way of obtaining the value of y for a given value of x. It states merely that a function exists when any way of determining y is known. In order to specify a definite function, a way of obtaining its values must be given. A particular function y, or $f(x)$, is said to be *defined* for a range of values of x when a rule or process is made known by which a value of y can be found when a value of x is given. This can be done in various ways, the most common of which will now be illustrated.

4.1. Functions defined by expressions. For the purpose of algebra, the most useful way of defining a function of x is by an expression or formula involving x. The following are examples:

(*a*) $3x+2$, x^2, 10^x define functions of x for all values of x since, for any number assigned to x, each expression takes a corresponding value.

(*b*) $\dfrac{x^2+2}{x-1}$ defines a function of x for all values of x except $x=1$. The value $x=1$ must be excluded since, for this number, the given expression would require the division of 3 by 0. *The ordinary process of division fails when the divisor is 0.* A special definition for division by 0 might be made if the consequences of such a definition were useful. There is, however, no such useful definition and hence division by 0 is not defined. A fractional form with 0 for denominator is therefore to be rejected as having no meaning.

(*c*) Since no real number has a square which is negative, hence $\sqrt{x}$* defines a function of x only when $x \geqq 0$; similarly, $\sqrt{x-2}$ defines a function of x for $x \geqq 2$.

(*d*) $\sqrt{1-x^2}$ defines a function of x for $|x| \leqq 1$, that is, for $-1 \leqq x \leqq 1$.

In our nomenclature, it is a matter of indifference whether we speak, for example, of the function x^2+1 or the function defined by the expression x^2+1. The shorter form is the one customarily used.

4.2. Functions defined by graphs. Suppose that y, or $f(x)$, is a function of x defined by an expression as in 4.1, and that values of y have been determined corresponding to several values of x. From these corresponding values of x and y, points may be plotted on a graph and a curve drawn through these points. This curve has the property that the abscissa and ordinate† of any point on it are values of x and y which correspond according to the functional relation $y=f(x)$. If, for example, the function is $y=\sqrt{25-x^2}$, the graph (see fig. 3) is the upper half (with its end points) of the circle with radius 5 and centre at the origin. In this case, either the expression or the semicircle may be taken as the definition of the function $f(x)$ since from either it is possible to find the value of y corresponding to a given value of x.

---

*$\sqrt{a}$ will be used to denote the positive square root of a, $-\sqrt{a}$ to denote the negative root. Thus, $\sqrt{4}=2$, not ± 2, and $\sqrt{a^2}=|a|$.

†When, in the use of graphs, a point is represented by (x, y), x and y are called the coordinates of the point, x being the abscissa and y, the ordinate.

Again, suppose that a graph only is given. Corresponding to any abscissa x within a proper range, an ordinate y is determined by the curve, and hence the graph serves as a definition of a function. If, for example, the graph is a straight line parallel to the axis of x, the function defined is a constant. The barometric chart of fig. 2 shows a curve traced during one week by a self-registering barometer. This curve serves to define the atmospheric pressure P as a function of the time, t. In such a case, no expression for the function is available.

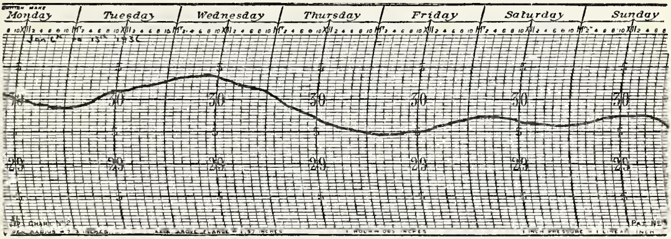


Fig. 2

A graph portrays in a vivid manner the relationship of two variables. In order that the graph shall convey its whole meaning, it should be regarded not merely as a number of points joined by a smooth curve but rather as a continuous record of the way in which changes in the independent variable bring about changes in the function. As he runs his pencil along the curve, the student should observe when the function is increasing and when decreasing, when it is changing slowly and when rapidly, when it is greatest and when least; he is advised to examine fig. 2 from this point of view. Again, in fig. 3, the graph shows that as x increases from -5 to 0 the function increases, at first rapidly and then more slowly, from 0 to 5, its greatest value; then, as x continues to increase from 0 to 5, the function decreases, at first slowly and then more rapidly, from 5 to 0.

4.3. Functions defined by tables of values. When a function $y=f(x)$, has been defined either by an expression or by a graph, a table of values can be made, showing values of x with the corresponding values of y. The following example of a function

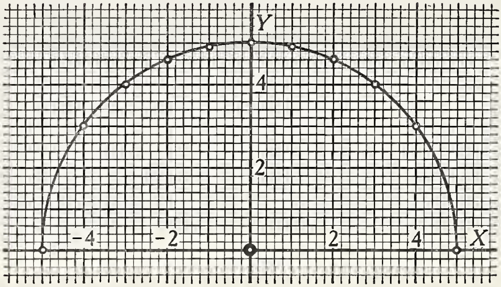


FIG. 3

Table of values

Expression
$y=\sqrt{25-x^2}$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	0	3	4	4.6	4.9	5	4.9	4.6	4	3	0

shows the graph, the expression, and a table of values, which might be further extended. It is by means of such tables of values that many functions are used in numerical work. The function $\log x$ and the trigonometric functions, $\sin x$, $\cos x$, $\tan x$, etc., are common examples. In such cases, the table of values may be used as a definition of the function. Thus a logarithmic table might be taken as a definition of $\log x$, or a table of sines as a definition of $\sin x$.

Cases occur in which a table of values of x and y is known, but no expression giving the relation between them is available. This table then constitutes the original definition* of a function $f(x)$. While this is not the most satisfactory way of defining a function, it has to be used in some cases. This applies, for example, to tables obtained from statistical records or from laboratory experiments. The mortality tables used by insurance companies furnish an example of a functional relation. The following extract from such a table shows, out of 100,000 people who reach age 10, the number surviving at each of the ages 11 to 21. Here y, the number of people surviving at each age, is a function of the age x.

x	y	x	y
10	100,000	16	95,550
11	99,251	17	94,818
12	98,505	18	94,089
13	97,762	19	93,362
14	97,022	20	92,637
15	96,285	21	91,914

For any function, we may form both a table of its values and a graph. The table may then be thought of as a skeleton of the graph and the graph as the clothed body of the table. A student should accustom himself to interpret either as a record of the way in which one variable quantity depends on another. It has already been pointed out that the nature of this dependence is obscured by regarding the graph merely as a number of points. For the same reason, the table should be regarded as something more than an array of isolated numbers. One should accustom himself to observe how these numbers are changing from one to another. For example, if the eye is allowed to run across the table of fig. 3, it is observed that, as x increases from -5 to 5, y increases from 0 to 5 and then decreases from 5 to 0. Moreover, the table shows that equal changes in x do not always produce equal changes in y; when x increases from -5 to -4, y increases by 3 units, but when x increases from -4 to -3, y increases by only 1 unit. The rate of change of y with respect to x is seen to be most rapid for values of x at the ends of the table. Similarly, the above extract from a mortality table should be read so as to make clear what happens to the number of survivals as age advances. Do equal increases in age result in equal numbers of deaths?

---

*When a function is defined by a table of values the following question arises: Is the range of values of the independent variable restricted to the values which appear in the table or can values intermediate between these be taken? It is easy to see that both cases are possible. In the mortality function described above, x is not restricted to integral values although the table contains only integers. In the function of exercise 19, page 20, the independent variable n can take only integral values.

4.4. Functions defined by statements. A statement in words is a common way of defining a function, and for some functions it has advantages over any of the three ways already mentioned. The usual definition of the logarithm is a case in point. The following examples are added for further illustration:

(*a*) a certain function $f(x)$ is defined to have the value 1 when x is an integer, and the value 0 when x is not an integer;

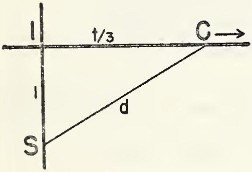
(*b*) a person's income tax, as a function of his income, is defined by statements which are printed on the income-tax forms;

(*c*) a customs duty, as a function of the invoice price of an article, may be defined by statements on the appropriate official forms.

It may happen that the description of a function does not specify precise values but indicates merely the nature of its variation. The general character of the function can then be shown by a graph, even though definite points through which the graph passes are not known. If, for example, you had recorded during the past month which days were hot, warm, cool, etc., you could then sketch a graph showing the variation of temperature during the month. Such a sketch aims to show the "ups and downs" of the function, rather than its precise values.

Example 1. Two straight roads intersect at right angles. From a service station 1 mile from the intersection on one of the roads, a car travelling on the other road at 20 miles an hour is observed to pass the intersection. Find the function of t which gives the distance of the car from the service station t minutes later.

Let S, I, and C (see figure) denote the positions of the service station, the intersection, and the car. Then S and I are fixed, while C moves on the line IC. The distance IC, t minutes after the car was at I, is $t/3$ miles. The required distance d is found from the Pythagorean relation. We have



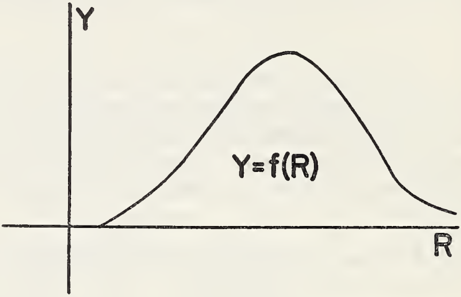
$$d^2=1+t^2/9=(9+t^2)/9$$

and $$d=\sqrt{9+t^2}/3,$$

which is the required function.

Example 2. Farm crops require a proper amount of rainfall for a good yield. Some seasons are too wet and others too dry. Assuming that the rain occurs uniformly during the growing season, sketch a graph to indicate the dependence of yield on rainfall.

Denote the yield by Y and the amount of rainfall by R, so that $Y=f(R)$. If there were no rainfall there would be no crop; hence the required graph goes through the origin. Since a certain minimum of rainfall is required to produce a crop, the graph follows the axis of R for some distance. After this, the graph rises to a maximum for a value of R which might be called the *optimum* rainfall and then falls again. The general character of the graph is shown on the figure.



Example 3. For the function $\frac{x+2}{x-2}$, construct a table of values and a graph and discuss the character of the function.

The given expression defines a function for all values of x except $x=2$. Hence it is of interest to consider particularly the behaviour of the function when x is close to 2. In the following table, values of x near 2 have, accordingly, been chosen closer together than in the rest of the table. The calculations are simplified by writing the expression in the form $1+\frac{4}{x-2}$.

x	-4	-3	-2	-1	0	1	$\frac{3}{2}$	$\frac{7}{4}$	2	$\frac{9}{4}$	$\frac{5}{2}$	3	4	5	6
y	$\frac{1}{3}$	$\frac{1}{5}$	0	$\frac{-1}{3}$	-1	-3	-7	-15	no value	17	9	5	3	$\frac{7}{3}$	2

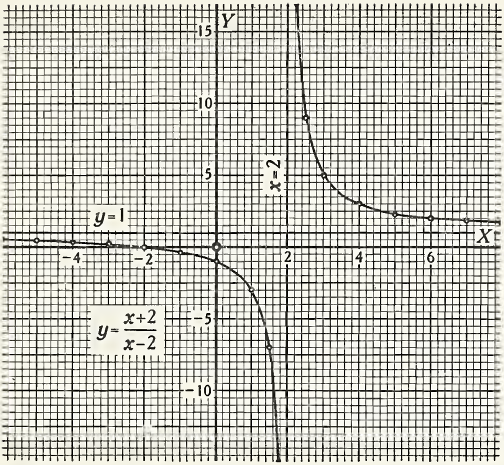


FIG. 4

The graph, fig. 4, drawn from the table, pictures the behaviour of the function for changes in x. Thus as x increases (algebraically) from large negative values, the function is at first slightly less than 1 and slowly decreasing. This decrease becomes more rapid as x approaches 0 and exceedingly rapid as x nears 2. When x has passed the exceptional value 2, y, starting with very large values, decreases with great rapidity and then more slowly, finally dropping closer and closer to 1. The graph consists of two symmetrical parts, each of which approaches more and more closely to the lines $x=2$ and $y=1$, for large values of y and x respectively.

EXERCISES

1. If the mth day of March is the nth day of the year, express n as a function of m (*a*) if the year is not a leap year, (*b*) if the year is a leap year.

2. One side of a right-angled triangle is x and the hypotenuse is 5. Express as a function of x (*a*) the third side, (*b*) the area of the triangle.

3. A rectangle of area 100 has one side of length y. Express as a function of y (*a*) the other side, (*b*) the diagonal.

4. A cylinder with a fixed volume V has variable radius and height. Express (*a*) its height as a function of its radius, (*b*) its radius as a function of its height.

5. The specifications for a rectangular room call for a square floor and a volume of 1600 cu. ft. Express (*a*) the height as a function of the length of a side, (*b*) the diagonal (from a corner of the floor to the opposite corner of the ceiling) as a function of the side, (*c*) the diagonal as a function of the height.

6. With 120 rods of wire fence a farmer makes a rectangular enclosure along the side of a field which is already fenced. Express the area of the enclosure as a function of each of its two unequal sides.

7. Given that 1 in.=2.54 cm., express the number of metres (m) in an arbitrary length as a function of the number of yards (y).

8. Express the amount A of \$100 at 5% per annum, simple interest, for m months, as a function of m.

9. (*a*) Express the number of degrees Fahrenheit (F) in an arbitrary temperature as a function of the number of degrees Centigrade (C); (*b*) Express C as a function of F.

10. A boy $4\frac{1}{2}$ feet tall walks away from a lamp-post on which is a light 12 feet from the ground. Find the length of his shadow (l) as a function of his distance (d) from the post.

11. Find an expression for the function $f(x)$ which is defined by the following table of values:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-7	-5	-3	-1	1	3	5	7	9

12. Find an expression for the function $f(x)$ which is defined by the following table of values:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	20	13	8	5	4	5	8	13	20

13. Construct a table of values and draw the graph of the function $x+\frac{1}{x}$. In which quadrants does the graph lie? Explain the behaviour of the function for numerically small and for numerically large values of x.

14. The atmospheric pressure is a function of the altitude above the earth. (Consider the temperature constant.) As the altitude increases how does this function change? For equal increases of altitude are there equal changes of pressure? Indicate by a graph how you would judge the pressure to change with the altitude.

15. The temperature of Lake Ontario is around 39°F. in the coldest winter months and is around 65°F. in the warmest months of summer. There is a seasonal lag in the temperature change; that is, the lake reaches its maximum temperature about a month after June 21, the date of midsummer according to the sun. Show by a graph what you consider to be the nature of the function $T=f(t)$, where T is the temperature of the water and t, the time, ranges over a year.

EXERCISES ON CHAPTER I

A

1. If $f(x)=9-x^2$, find $f(3)$, $f(-3)$, $f(0)$, and $f(4)$.

2. If $f(x)=4x-x^2$, find $f(0)$, $f(2)$, $f(4)$, and $f(2-x)$.

3. If $f(x)=(3x+4)/(5x+2)$, find $f(0)$, $f(1)$, $f(-2)$, $f(-4/3)$, and $f(1/x)$.

4. State the range of values for which each of the following expressions defines a function of x:

$$(a)\ \frac{1}{x}+\frac{1}{x-4},\quad (b)\ \frac{1}{x^2-1},\quad (c)\ \sqrt{x^2-1},\quad (d)\ \frac{1}{\sqrt{x^2-1}}.$$

5. A triangle has a fixed base of 10 inches and a variable height of h inches. Express its area as a function of h.

6. A rectangle has a fixed perimeter of 12 feet and a variable height of h feet. Express as a function of h (*a*) its area, (*b*) its diagonal.

7. (*a*) Construct a table of values and a graph for the function $y=x^2$ for values of x from -5 to 5. (*b*) From the graph describe the behaviour of the function as x increases from -5 to 5. (*c*) From the expression x^2 or from the table tell what change in y takes place (i) when x is doubled, (ii) when x is halved. Is a change in y equal to the corresponding change in x? Do equal changes in x produce equal changes in y? At what part of the table is y changing fastest with respect to x?

8. Write an expression for the square of the reciprocal of x. Find the values of this function when

$$x=5,\ 4,\ 3,\ 2,\ 1,\ 1/2,\ 1/3,\ 1/4,$$

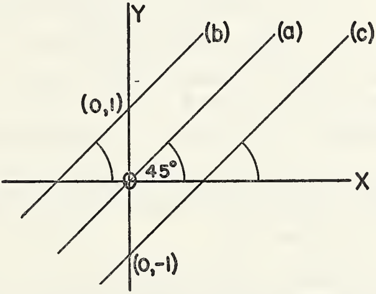
and draw the graph of the function. Describe the behaviour of the function as x gets close to 0.

9. Make a table of values of x^3 for integral values of x from -6 to 6. Tell, from this table, how y changes as x increases.

If x is doubled is y doubled? Does a unit change in x always produce the same change in y? At what part of the table does y change most rapidly with respect to x?

10. From a square of cardboard 12 inches wide a box is to be made in the following manner. A square of side x is cut from each corner of the cardboard, after which the sides are turned up through a right angle and fastened together to form the box. Express the volume of the box as a function of x.

11. Write expressions for the functions defined by the graphs (*a*), (*b*), and (*c*) in the accompanying figure.



12. Write an expression for the function defined by each of the following tables of values:

(*a*)

x	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$
y	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3

(*b*)

x	0	1/3	2/3	1	4/3	5/3	2	7/3	8/3	3
y	0	1	2	3	4	5	6	7	8	9

(*c*)

x	1	2	3	4	5	6
y	60	30	20	15	12	10

13. When a child rolls a snowball in soft snow the volume of the snowball increases more and more rapidly. Suppose the volume doubles for each revolution. If the original volume is v what is it after 5 revolutions? Sketch a graph of the volume V as a function of the number n of revolutions.

B

14. If $R(x)=a^x+a^{-x}$ and $S(x)=a^x-a^{-x}$, find the value of

$$\frac{\{R(2x)\}^2+\{S(2x)\}^2}{\{R(2x)+S(2x)\}^2}.$$

15. If $f(x)=1/(1+x^2)$, simplify $f(x)+f(1/x)+\{1/f(x)\}\cdot f(1/x)$.

16. If $F(x+1)=x^2+x+1$, find $F(y)$, $F(x)$, and $F(x-1)$.

17. State the range of values for which each of the following expressions defines a function of x:

$$(a)\ \sqrt{x^2+16},\quad (b)\ \sqrt{x^2-16},\quad (c)\ \sqrt{16-x^2},\quad (d)\ 1/\sqrt{16-x^2}.$$

18. Construct a graph of the function $y=10/x$. Describe the behaviour of the function for numerically small and for numerically large values of x.

19. If the thickness of each page of a book is 0.005 inches and that of each cover is 0.125 inches, express, as a function of n, the total thickness of a book of n pages. Does doubling the number of pages double the thickness of the book?

20. Write an expression for the function which is the sum of 5 terms containing in order the 5 integral powers of x from the first power to the fifth, each term having a coefficient equal to the reciprocal of the index of its power of x.

21. Find an expression for the function defined by the following table of values:

x	-4	-3	-2	-1	0	1	2	3	4
y	-10	-7	-4	-1	2	5	8	11	14

22. The gravitational attraction of two masses m_1 and m_2 at a distance r apart is $F = km_1m_2/r^2$, where k is a constant (Newton's Law of Gravitation). What is the effect, on the attraction, of (*a*) doubling the distance between the bodies, (*b*) diminishing the distance to 1/10 of its former value?

23. If $R(x)$ stands for the reciprocal of x, show that

$$R(x)+R(y)=R\left(\frac{xy}{x+y}\right).$$

24. If $S(x)$ stands for the square of x, show that

$$S(x+y)+S(x-y)=2\{S(x)+S(y)\}.$$

25. If $f(x)=p(x-a)+q(x-b)$, show that

$$p=\frac{f(b)}{b-a} \text{ and } q=\frac{f(a)}{a-b}.$$

26. Given $F=100/r^2$, make a table of values and draw the graph of the function.

27. For what values of x is x^3 equal to x^2? For what values is x^3 less than x^2 and for what values greater? Make a similar comparison of x^3 and x^4.

28. A truck with a light load makes a journey of 50 miles at v miles per hour and returns with a heavy load at two-thirds of this speed. Express as a function of v (*a*) the total time taken, (*b*) the average speed for the whole trip.

29. A body, starting from rest, falls s feet in t seconds, where s and t are observed to have the following corresponding values:

t	0	1	2	3	4	5
s	0	16	64	144	256	400

Construct a graph showing s as a function of t. From the graph estimate (*a*) the distance fallen in $3\frac{1}{2}$ seconds; (*b*) the time it takes the stone to fall 100 feet.

30. Given $f(-2)=6$, $f(-1)=0$, $f(-\frac{1}{2})=-3/8$, $f(0)=0$, $f(\frac{1}{2})=-3/8$, $f(1)=0$, $f(2)=6$, sketch a graph of $f(x)$ and from it estimate the value of $f(3/2)$ and of $f(-3/2)$.

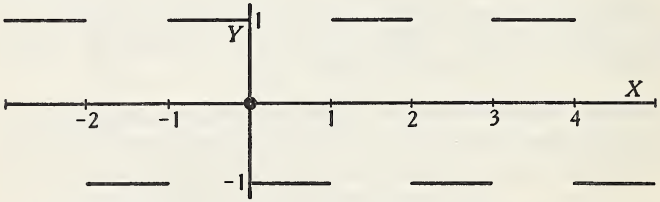
31. For the function $y=6-x-x^2$, construct a table of values and draw the graph. Describe the behaviour of the function as x decreases from 4 to -5.

32. On page 334 the function $A=(1+i)^n$ is tabulated. This table gives the amount of 1 dollar invested at compound interest at the rate i for n interest periods. Use the table to answer the following questions: If the number of interest periods is doubled, is the amount doubled? If the rate of interest is doubled, is the amount doubled? If the rate is doubled, is the compound interest doubled?

C

33. Write an expression for the function which is the sum of the square and the cube of x, diminished by the reciprocal of the square of the sum of x and the square of x.

34. Define in words the function which has the following graph. (The points marked on the x-axis form part of the graph.)



35. Draw the graph of the function which is defined as follows:

for $0 \leqq x < 1$, $y=x$; for $1 \leqq x < 2$, $y=2-x$;
for $2 \leqq x < 3$, $y=x-2$; for $3 \leqq x < 4$, $y=4-x$;
for $4 \leqq x < 5$, $y=x-4$; for $5 \leqq x < 6$, $y=6-x$; etc.

36. Construct the graph of the function defined as follows for positive values of x: the value of $f(x)$ is the greatest integer which is not greater than x.

37. If $f(x)=x^2-4x$, find the value of $\frac{f(x+h)-f(x)}{h}$. If h is very small, what is a good approximation to this value?

38. If $F(x)=3x+4$, express in terms of x the following functions:

(*a*) $F\{F(x)\}$; (*b*) $F\{F(1/x)\}$; (*c*) $F\{1/F(x)\}$.

39. Construct a table of values and draw the graph of $y=|x|$.

40. The period of oscillation of a simple pendulum is given approximately by the formula $T=2\pi\sqrt{l/g}$, where l is the length of the pendulum and g is a constant. Taking $g=32$, construct a table of values and draw the graph of the function of l. Does an increase in l always produce an increase in T? If l is doubled, is T doubled? What is the effect on T when (i) l is quadrupled, (ii) l is increased 9-fold?

41. A pint of cold water in a kettle is placed on a gas burner. In 10 minutes the water comes to a boil and in 15 minutes more the kettle has boiled dry. If y measures the amount of water in the kettle t minutes after it was placed on the burner, describe and graph the function $y=f(t)$ for the range of t from 0 to 30.

42. If, in exercise 41, the gas is at first turned on low, then after the water has boiled for 5 minutes the gas is turned on full, show how the graph of y will be affected.

43. If, in exercise 41, T denotes the temperature of the water t minutes after it was placed on the burner, make a graph of T as a function of t for the range of t from 0 to 25.

44. If, in exercise 43, the gas is turned on low for the first 5 minutes after the water is placed on the burner and then turned on full, show how the graph of T will be affected.

45. A moth-ball evaporates in such a way that its diameter diminishes $\frac{1}{4}$ inch in each 6-month interval. If the ball was originally 1 inch in diameter, draw graphs to show the radius, the surface area, and the volume as functions of the time.

46. A vat contains a saturated solution of salt, used for tanning. It is desired to wash out the salt without diminishing the amount of liquid in the vat. For this purpose fresh water is let into the vat while, at the same time, the solution runs out of a tap at the bottom. The liquid is kept stirred so that the solution is nearly homogeneous throughout. Are equal amounts of salt washed out in equal times? Will the last particle of salt ever be washed out of the vat? Indicate by a graph your idea of the amount of salt in the vat as a function of the time.

CHAPTER II

THE LINEAR FUNCTION AND APPLICATIONS

5. Definition and examples. *The function* $ax+b$, *where* a *and* b *are constants and* $a\neq 0$, *is called a* ***linear integral function,*** *or more briefly a* ***linear function,*** *of* x. The following illustrations furnish examples:

(*a*) $3x$, $5t-4$, $\frac{7s+1}{4}$ are linear functions of x, t, and s, respectively.

(*b*) If the construction costs for a building are \$$b$ for materials and \$$a$ a day for labour, then the cost in dollars at the end of n days is $an+b$, a linear function of n.

(*c*) If a car starts from a point d miles west of a town and travels west at a uniform speed of v miles an hour, then its distance from the town at the end of t hours is $vt+d$ miles.

6. The graph of a linear function. We shall denote by y the function $ax+b$, where a and b are any constants. The function represented is then either a linear function (if $a\neq 0$) or a constant (if $a=0$). If we compute values of y corresponding to chosen values of x, we may use each pair (x, y) of corresponding values as the coordinates of a point, and construct from the resulting points a graph. It is a matter of indifference whether we speak of the graph of the equation $y=ax+b$, or of the function $ax+b$. To find the nature of this graph*, consider first the example $2x-3$. Corresponding values of x and y are given in the following table:

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*A student of analytic geometry will recognize the equation $y=ax+b$ as that of a straight line having slope a and y-intercept b.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
y	-11	-9	-7	-5	-3	-1	1	3	5	7	9

When the points are plotted (fig. 5) they appear to lie on a straight line. Is this a general property of a linear function? The answer is given in the following theorem.

Theorem. *The graph of $y=ax+b$ is a straight line, whatever be the values of a and b.*

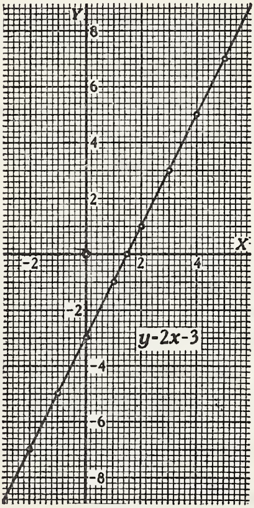


Fig. 5

Let P_1 and P_2 be any two points on the graph. Then their coordinates (x_1, y_1) and (x_2, y_2) satisfy the equation $y=ax+b$. We prove first that any third point P_3, whose coordinates satisfy the equation, lies on the straight line P_1P_2. For, we have

$$y_1 = ax_1+b,$$

$$y_2 = ax_2+b.$$

Hence $\quad y_2-y_1 = a(x_2-x_1),$

or
$$\frac{y_2-y_1}{x_2-x_1} = a.$$

By hypothesis, this result also holds if we replace (x_2, y_2) by (x_3, y_3).

Hence

$$\frac{y_3-y_1}{x_3-x_1} = a = \frac{y_2-y_1}{x_2-x_1}.$$

In fig. 6, in which the given points are represented, $y_2-y_1=MP_2$, $x_2-x_1=P_1M$, $y_3-y_1=NP_3$, $x_3-x_1=P_1N$. Hence,

$$\frac{NP_3}{P_1N}=\frac{MP_2}{P_1M}.$$

Since the angles at M and N are right angles, it follows that the triangles P_2P_1M and P_3P_1N are similar. Hence, the angles P_2P_1M and P_3P_1N are equal and the point P_3 lies on the straight line P_1P_2.

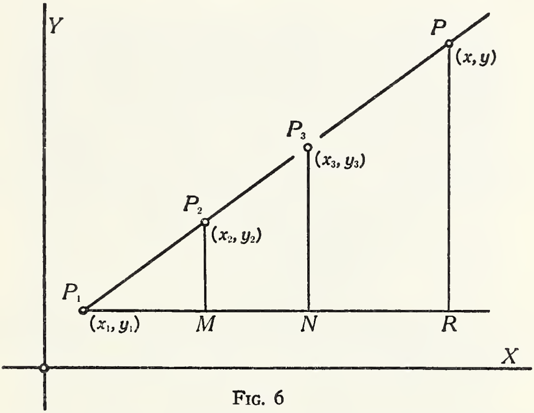


FIG. 6

It remains to find out whether every point on the line P_1P_2 satisfies the equation $y=ax+b$. Let P be any point (x, y) on the line. Then, from fig. 6,

$$\frac{RP}{P_1R}=\frac{MP_2}{P_1M}=a.$$

That is, $$\frac{y-y_1}{x-x_1}=a,$$

or $$y-y_1=ax-ax_1.$$

But $$y_1=ax_1+b.$$

Hence, by addition, $$y=ax+b,$$

and the point (x, y) satisfies the equation.

Since the equation $y=ax+b$ represents a straight line, it is called a *linear equation* in the two variables x and y. The equation $lx+my+n=0$, with $m\neq 0$, is also a linear equation since it may be written $y=-\frac{l}{m}x-\frac{n}{m}$. Moreover, if $m=0$ and $l\neq 0$, the equation reduces to $x=-\frac{n}{l}$, which is seen to denote a straight line parallel to the axis of y. Hence $lx+my+n=0$ is called a linear equation provided l and m are not both zero. The student should accustom himself to associate linear equations with the lines they represent. A few examples are shown in fig. 7.

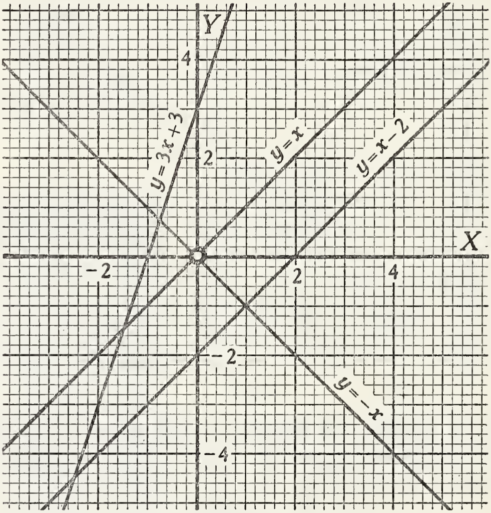


Fig. 7

6.1. The equation $ax+b=0$. The line $y=ax+b$, in which $a\neq 0$, is not parallel to the axis of x and therefore crosses it at some point. At this point y has the value 0, and hence the abscissa of the point is given by $ax+b=0$, that is, by $x=-b/a$. The fact that this equation has one and only one root is in accord with the fact that the line cuts the axis of x in just one point.

6.2. The relation between the changes in x and y when $y=ax+b$. Let us refer again to the graph or the table corresponding to the function $y=2x-3$ (fig. 5). If x increases by 1 unit, y increases by 2 units; if x increases by 2 (or 3) units, y increases by 4 (or 6) units, etc. Thus it is observed that equal changes in x produce equal changes in y, and that y changes twice as fast as x. Do similar properties hold for all linear relationships? The answer is furnished by the general linear function $y=ax+b$. Let x_0 and y_0 be corresponding values of x and y, so that $y_0=ax_0+b$. Now let x increase by any constant amount d, so that the new x is x_0+d. Then the new y is

$$\begin{aligned} a(x_0+d)+b &= ax_0+b+ad \\ &= y_0+ad. \end{aligned}$$

This shows that a constant change of d in x always produces a constant change of ad in y. Hence the change in y is a times that in x, which means that y changes a times as fast as x.

If y is a linear function of x, then equal changes in x always produce equal changes in y.

7. Determination of a linear function from two known values. The function $ax+b$ depends upon two arbitrary constants a and b. If values of the function are known for two given values of x, then values of a and b, and hence the linear function, can be found.

EXAMPLE. Find the linear function of x which takes the value 1 when $x=2$, and the value 13 when $x=4$.

In order that $ax+b$ satisfy these conditions, we must have

$$\begin{aligned} 2a+b&=1, \\ 4a+b&=13, \end{aligned}$$

whence $a=6$, $b=-11$, and the required function is $6x-11$.

By the same method, we may find the linear function $ax+b$ which takes the values m and n when x equals c and d, respectively. In this case,

$$a=\frac{m-n}{c-d}, \qquad b=\frac{cn-dm}{c-d},$$

and the linear function is

$$f(x)=\frac{m-n}{c-d}x+\frac{cn-dm}{c-d}.$$

This result may be expressed in the convenient formula

$$f(x)=\frac{x-d}{c-d}m+\frac{x-c}{d-c}n,$$

which shows at once that $f(c)=m$ and $f(d)=n$. The details of the work are left as an exercise for the reader. (See exercise 31, page 46).

The student of analytic geometry will recognize that the problem of determining a linear function which has given values for two given values of x is one aspect of the problem of finding the equation of a straight line through two points.

EXERCISES

1. Which of the following expressions are linear functions of x and which are not? Give your reason for each answer.

(*a*) $3x+8$ (*b*) $8-3x$ (*c*) $3x$

(*d*) $\dfrac{1}{ax+b}$ (*e*) a^x+b (*f*) ax^2+b

(*g*) $\frac{5x-2}{5x+2}$ (*h*) $\frac{x^2-4}{x-2}$ (*i*) $-\frac{3}{4}x-5$

(*j*) $3(2x^2+5)-6(x^2-2x)$ (*k*) $\frac{3x^2+7}{x}$

2. Give the general form of a linear function of x which is zero when x is zero.

3. Given that $f(x)=0.56-0.032x$, find (*a*) the value of $f(4.7)$, (*b*) the value of x which will give the function the value 0.32.

4. Find the linear function of r which has the values -16 and 8 when r equals -2 and 2, respectively.

5. Find the linear function of t which has the values 1.2 and 6 when t equals -1 and 1, respectively.

6. Find the linear function of x which has the values -1 and 6 when x equals -1 and 1, respectively.

7. Find the linear function $f(y)$, given that $f(2)=0.8$ and $f(5)=5.6$.

8. Find the linear function $f(z)$, given that $f(2)=17$ and $f(4)=-1$.

9. A linear function of x has the value 2 when $x=2$, and the value -1 when $x=3$. Find its value when $x=4$.

10. Show that a linear function of z exists which takes the values 3, 5, 7, when z has the values 1, 2, 3, respectively. Find the function. How much of the data is necessary?

11. Does the following table of values suggest a linear relation between x and y? If so find the relation.

x	-3	-2	-1	0	1	2	3
y	11	8	5	2	-1	-4	-7

12. What change in the function $3x+8$ is produced by (*a*) a change of 2 in x? (*b*) a change of d in x?

13. A variable y depends on x in such a way that $y=10$ when $x=2$ and an increase of 1 in x always produces an increase of 7 in y. What linear function will satisfy these conditions?

14. A number of 2-inch planks are piled on a platform which is $1\frac{1}{2}$ feet above the ground. Show that the height, in inches, of the top of the pile above the ground is a linear function of the number of planks. Draw the graph of the function.

15. A ship leaves port and travels in a straight line with constant speed, v. Show that its distance from port is a linear function of the time it has travelled. Is this true if the velocity varies? Is it true if the ship travels in a curved path?

16. Show that the number of minutes after 6 o'clock is a linear function of the number of hours after 4 o'clock.

17. A meter cab charges 15c for the first quarter-mile and 5c for each quarter-mile thereafter. Find the cost of travelling m miles, as a function of m.

18. Show that the number of right angles in the sum of the angles of a polygon is a linear function of the number of its sides. If the number of sides of a polygon increases at a certain rate, state precisely the manner of change of the number of right angles in the sum of its angles. Does an increase of 2 in the number of sides always produce the same increase in the number of right angles?

19. Draw the graphs of $y=x+b$ for several values of b. What property have all the graphs?

20. Draw the graphs of $y=ax$ for several values of a. What property have all the graphs?

8. Direct variation. The line denoted by $y=3x$ passes through the origin. Since at any point on this line $y/x=3$, the variables x and y preserve a constant ratio. For a change of c in x, there is a change of $3c$ in y; either variable is said to vary directly as the other.

Definition. *y is said to vary directly as x when $y=kx$, k being a constant different from zero.*

The definition can be stated in the alternative form: *y varies directly as x when the ratio y/x is a constant, not zero.*

For brevity, the word *directly* is often omitted and y is said to vary as x. If y varies as x and a pair of corresponding values of x and y are known, the constant k can be determined. This corresponds with the fact that a line through the origin is determined if one other point on the line is known.

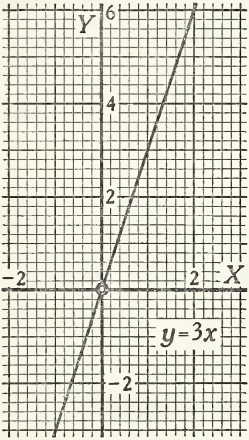


Fig. 8

Example 1. The area of a square inscribed in a circle varies as the area of the circle. When the area of the circle is 15.7, the area of the square is 10. Find the area of the square when the area of the circle is 20.

Using C and S to denote the two areas, we have

$$S=kC.$$

Since $S=10$ when $C=15.7$,

therefore $10=k\,(15.7)$,

and

$$k=\frac{10}{15.7}.$$

Hence $$S=\frac{10}{15.7}\,C.$$

When $C=20$,

$$S=\frac{200}{15.7}=12.7.$$

Cases of two variables x and y arise in which y varies as the square of x, or as some higher power. Again, y may vary as the reciprocal of x; in this case, $y=k/x$ and y is said to vary *inversely* as x. Similarly, if $y=k/x^2$, y varies inversely as the square of x. Further illustrations will occur to the reader.

EXAMPLE 2. The attraction of a planet on a body of given mass varies inversely as the square of the distance of the body from the centre of the planet. Compare the weight of a body at the surface of the earth with that of the same body at a distance of three times the earth's radius from its centre.

The weight of the body measures the force of attraction of the earth on it. If this force is F and the distance of the body from the earth's centre is x, we have $F=k/x^2$, where k is constant. When $x=r$, the earth's radius, let $F=F_1$. Then

$$F_1=\frac{k}{r^2},$$

and

$$k=F_1r^2,$$

whence

$$F=\frac{F_1r^2}{x^2}.$$

Now when $x=3r$,

$$F=\frac{F_1r^2}{9r^2}=\frac{F_1}{9}.$$

Hence, for example, a body which weighs 180 lb. at the surface of the earth would weigh 20 lb. if removed from the surface a distance equal to the diameter of the earth.

9. Joint variation. *If x and y are variables and z varies as the product xy, so that $z=kxy$, then z is said to vary jointly as x and y.*

Thus, for example, the area A of a triangle varies jointly as its base b and its height h, since $A=\frac{1}{2}bh$. Again, the simple interest I on a sum of money P varies jointly as the time t and the rate r, since $I=Prt$.

If z varies jointly as x and y, $z=kxy$. Then, when y is constant, z varies as x; also, when x is constant, z varies as y. For example, the area of a triangle varies as the base when the height is constant, and as the height when the base is constant.

We prove next the converse of this proposition. *If z is a function* of x and y which varies as x when y is constant and as y when x is constant, then z varies jointly as x and y when both x and y vary.* The steps of the proof are illustrated by fig. 9, in which x, y, and z represent respectively the base, the altitude, and the area of a triangle. However, the proof does not depend on this illustration.

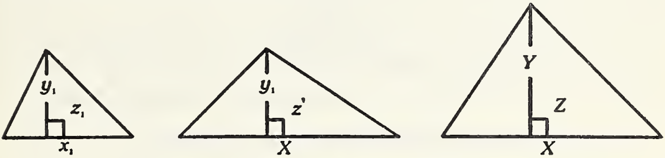


FIG. 9

Suppose that $z=z_1$ when $x=x_1$ and $y=y_1$, and that $z=Z$ when $x=X$ and $y=Y$. Let y keep the value y_1 while x changes from x_1 to X. During this change z varies as x, hence $z=kx$, where k is constant. Hence, if the resulting value of z is z', then $z_1=kx_1$ and $z'=kX$, from which

$$\frac{z'}{z_1}=\frac{X}{x_1}.$$

Now let x keep the value X while y varies from y_1 to Y. Then z changes from z' to Z and, as in the former case,

$$\frac{Z}{z'}=\frac{Y}{y_1}.$$

It follows, by multiplication, that

$$\frac{Z}{z_1}=\frac{X}{x_1}\cdot\frac{Y}{y_1},$$

---

*The definition of functions of two variables is given in chapter V. A discussion of such functions is not necessary for the understanding of the present theorem.

and $$Z=\frac{z_1}{x_1y_1}XY.$$

Since $\frac{z_1}{x_1y_1}$ is a constant, K, and X, Y, Z are any corresponding values of x, y, z, we have $z=Kxy$, which proves the proposition.

Example 1. The force of the wind on a flat surface (such as a sail or the windshield of a car) varies jointly as the area of the surface and the square of the velocity of the wind. If a gale of 50 miles an hour produces a force of 160 pounds on an area of 20 square feet, find (*a*) the force produced on 10 square feet by a gale of 60 miles an hour, (*b*) the wind velocity necessary to produce a force of 256 pounds on an area of 50 square feet.

The given data are expressed by the equation

$$F=kAv^2.$$

When $A=20$ and $v=50$, $F=160$;

hence $$160=k\cdot 20\cdot 50^2, \text{ and } k=\frac{16}{5000}.$$

Therefore $$F=\frac{16}{5000}Av^2.$$

(*a*) Now when $A=10$ and $v=60$,

$$F=\frac{16}{5000}\cdot 10\cdot 60^2=115.2.$$

The required force is 115.2 pounds.

(*b*) When $A=50$ and $F=256$,

$$256=\frac{16}{5000}\cdot 50\cdot v^2,$$

whence $$v^2=\frac{5000}{16\cdot 50}\cdot 256=1600.$$

Hence $v=40$, and the required velocity is 40 miles an hour.

EXAMPLE 2. If y varies as x, show that x^2+y^2 varies as xy.

Since y varies as x, then $y=kx$ where k is a constant. In the ratio of the given functions substitute $y=kx$.

We get

$$\frac{x^2+y^2}{xy}=\frac{x^2+k^2x^2}{kx^2}=\frac{1+k^2}{k}.$$

Since this ratio is constant, it follows that x^2+y^2 varies as xy.

EXERCISES

1. Express by formulas the following statements:

(*a*) A varies as the square of r;
(*b*) A varies inversely as the square of r;
(*c*) m varies as the cube of n;
(*d*) y varies as the cube root of x;
(*e*) z varies directly as r and inversely as the square of s;
(*f*) z varies as the sum of x and y;
(*g*) u varies directly as the square root of y and inversely as the cube of z;
(*h*) r varies jointly as the square of p and the cube of q;
(*i*) z varies directly as the product of x and y and inversely as the *nth* power of t.

2. Use the language of variation to express the following formulas, after the manner of ex. 1: (*a*) $z=kx/y$; (*b*) $y=k/x^2$; (*c*) $z=kx/\sqrt{y}$; (*d*) $u=kx^2y$; (*e*) $y=k\sqrt[3]{x^2}$.

3. If u varies as v and $u=7$ when $v=2$, find the relation connecting u and v.

4. If y varies as x and has the value 6.5 when $x=4$, find its value when $x=7$.

5. If z varies jointly as x and y and $z=9$ when $x=2$ and $y=6$, find z when $x=y=4$.

6. The volume of a rectangular box varies jointly as its length, width, and height. What is the effect, on the volume of the box, of doubling its height and trebling its length?

7. A variable y is the sum of two terms of which one is constant and the other varies as x. When $x=2$, $y=11$ and when $x=4$, $y=17$. Find y as a function of x.

8. For an electric circuit with fixed voltage the current varies inversely as the resistance. If the current is 2 amperes when the resistance is 3 ohms, find the current when the resistance is 4 ohms; 9 ohms; 12 ohms.

9. The stretch of a spring varies as the weight applied. If a weight of 6 pounds stretches a spring 4 inches, how much will a weight of 8 pounds stretch it? What weight will compress the spring 3 inches?

10. The speed of a falling body varies as the square root of the distance fallen. If $v=80$ ft. per sec. when $s=100$ ft., find the formula for the speed. Find v when $s=400$ ft.

11. The mass of a metal cone varies jointly as its height and the square of the radius of its base. A certain cone has a mass of 80 oz. Find the mass of a second cone with 3 times the height and twice the base radius of the first.

12. If a three-dimensional figure varies in size but not in shape, its volume varies as the cube of any linear dimension. A pear-shaped balloon which measures 20 feet from top to bottom is inflated with 1600 cubic feet of gas. How much gas will be required to fill a similar balloon which measures 30 feet from top to bottom?

13. When an electric current flows through a wire, the amount of heat produced varies jointly as the resistance of the wire and the square of the amount of current. The resistances of two wires are in the ratio 4:9. What current passed through

the second wire will produce the same amount of heat as a current of 4 amperes through the first wire?

14. The volume of a mass of gas varies directly as its absolute temperature and inversely as its pressure. A mass of gas which occupies 1000 cu. cm. at 240° (absolute) and 800 mm. pressure has its temperature raised to 300° and its pressure lowered to 600 mm. Find the resulting volume.

15. If x varies as y, show that $2x^2-3xy$ varies as $4xy+y^2$.

16. If x varies as y, which of the following statements are true and which false?

(*a*) x^2 varies as y^2; (*b*) $1/x$ varies as $1/y$;
(*c*) 2^x varies as 2^y; (*d*) $\sqrt{x}$ varies as $\sqrt{y}$;
(*e*) $x+1$ varies as $y+1$; (*f*) $x-y$ varies as $x+y$;
(*g*) $x/2$ varies as $y/3$; (*h*) $\log x$ varies as $\log y$.

★10. The linear function used for interpolation. When a function $f(x)$ is defined by a table of its values, it is frequently necessary to find the value of $f(x)$ for a value of x which is intermediate between two values of x given in the tables. The process by which this is done is known as ***interpolation.*** Thus, if temperatures are recorded for each hour during the day, one might wish to interpolate the temperatures corresponding to the half-hours or quarter-hours. Again, if the average weights of a group of boys are given corresponding to heights of 36 inches, 38 inches, 40 inches, etc., one might interpolate the weights corresponding to heights of 39 inches, 38.5 inches, etc.

The simplest method of interpolation is that which makes use of two known values of a function, corresponding to two given values of the independent variable, and replaces the function between these values by a linear function. The manner in which changes in a linear function, y, are related to the corresponding changes in the independent variable, x, has been pointed out in sec. 6.2. What is needed for the present discus-

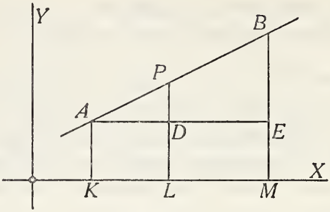


FIG. 10

sion is recalled by means of the diagram, fig. 10. Let A, P, and B be three points on the graph of a linear function. The values of the function at A, P, B are represented by AK, PL, BM, respectively, and the corresponding values of the independent variable by OK, OL, OM, respectively. Therefore, increases of AD, AE, in the value of the independent variable at A give rise to increases of PD, BE, in the value of the function.

From similar triangles, we have $PD : BE = AD : AE$. Hence, if the values of x are known for A, P, and B, and the values of the function are known for A and B, this relation enables us to determine PD. By adding PD to the value of the function at A, we obtain the value of the function at P. Similarly, if the values of y are known for A, P, and B and the values of x for A and B, the same relation enables us to determine AD. By adding AD to OK, the value of x at A, we obtain the value of x for which the function has the value PL.

This method is sometimes called the method of proportional parts. Since it makes use of a linear function, it is also called linear or straight line interpolation. To be sure, the function for which an interpolated value is wanted is not in general a linear function. It is, however, represented by a smooth curve and the principle involved is that of approximating to a short arc of a smooth curve by means of the corresponding chord. In fig. 11(i), it is seen that the interpolated value PL is slightly smaller than the actual value CL of the function, while in fig. 11(ii) the interpolated value is too large.

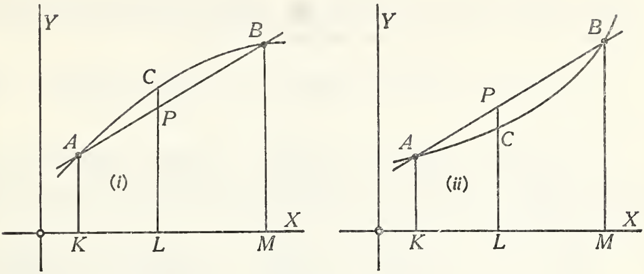


FIG. 11

Linear interpolation is employed constantly in making use of logarithmic and trigonometric tables as also in the tables used by financial houses and by statisticians.

EXAMPLE. Given log 21.84 $=1.33925$
and log 21.85 $=1.33945$,
find log 21.843.

Here an increase of .01 in the independent variable produces an increase of .00020 in the function. Therefore, an increase of .003 in the independent variable will produce an increase of

$$\frac{.003}{.01} \text{ of } .00020 = .00006$$

in the function. Thus

$$\log 21.843 = 1.33925 + .00006 = 1.33931.$$

EXERCISES

1. Given log 323.5=2.50987, log 323.6=2.51001, find log 323.55.

2. Given log 21.01=1.32243, log 21.02=1.32263, find the number whose logarithm is 1.32251.

3. Given that 66°=1.15774 radians and 67°=1.17519 radians, find the number of radians in 66°24′.

4. Given sin 36°28′= 0.59436, sin 36°29′= 0.59459, find sin 36°28.6′.

5. Given the following records of hourly temperatures, find, by linear interpolation, the temperature at 8.30 a.m. and at 10.30 a.m.

time	8 a.m.	9 a.m.	10 a.m.	11 a.m.
temperature	60°F.	63°F.	64.5°F.	65.5°F.

6. From the table on page 11, estimate the number of people surviving at age 15 years, 8 months.

EXERCISES ON CHAPTER II

A

1. Find the linear function of x which has the values 1/6 and 3/2 when x equals 1 and 3 respectively.

2. Find the linear function of t which has the values 15.5 and —4.5 when t equals —2 and 3 respectively.

3. Given that $y=5$ when $x=2$ and $y=12$ when $x=5$, find the value of y when $x=4$, assuming one of the variables to be a linear function of the other.

4. A linear function of t varies from 15 to 6 when t varies from 5 to 11. For what value of t does the function vanish?

5. A linear function of r decreases from 43 to 23 when r increases from 1 to 5. For what value of r is the function equal to r?

6. A linear function of x increases twice as fast as x and vanishes when x is zero. What is the function?

7. Show that the following pairs of values satisfy a linear relation between x and y. Find the relation.

x	-6	-4	-2	0	2	4	6
y	-29	-21	-13	-5	3	11	19

8. Of the expenses of a banquet, a part is constant and the remaining part varies as the number of guests. Show that the expenses are a linear function of the number of guests. Find an expression for this function if the cost for 40 guests is $80 and for 100 guests, $125.

9. The equation $c=6r$ gives approximately the relation between the circumference and the radius of a circle. If the radius of a circle is increased by 5, what is the approximate change in its circumference? Considering the equator of the earth as a circle, imagine another circle one foot above it all the way around the earth. Approximately how much longer is the second circle than the equator?

10. The pressure within a given mass of gas varies inversely as its volume. A certain mass of gas exerts a pressure of 21 pounds per square inch when its volume is 50 cu. in. What pressure will it exert when it occupies 60 cu. in.? What volume will it occupy when it exerts a pressure of 720 pounds per square foot?

11. The weight of a ball of metal varies as the cube of its radius. If a ball of radius 2 inches weighs 9.7 pounds, find the weight of a ball of the same metal with radius 5 inches.

12. An approximate formula for a person's weight is

$$w=5.5(h-60)+110,$$

where w is the number of pounds in his weight and h the number of inches in his height. Check the formula against your own weight. What increase of weight is to be expected for each inch of increase in height?

13. Since the function of ex. 12 is linear, its graph is a straight line. Construct this graph by first locating two points on it.

14. Given that y varies inversely as the square of x and that $y=1$ when $x=1$, construct a table showing values of y corresponding to $x=5, 4, 3, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. Draw a graph of the function for positive values of x. Describe the behaviour of the function as x approaches zero and as x becomes large. Describe the change in y when x is doubled, when x is halved, and when x is trebled. At what part of the graph is y changing most rapidly with respect to x? At what part is it changing most slowly?

B

15. If y is a linear function of x, show that x is a linear function of y.

16. Find the linear function of x which equals 6 when $x=6$ and which decreases by 11 when x increases by 2.

17. The following are 5 values of a function $f(t)$: $f(1)=-2$, $f(3)=4$, $f(4)=7$, $f(6)=15$, $f(9)=22$. What change in one of these numbers would make the values belong to a linear function?

18. Show that the following corresponding values of x and y suggest that y is approximately a linear function of x. Find the approximate linear function.

x	0	1	2	3	4	5
y	7.5	6.0	4.4	3.0	1.6	0.1

19. Find the linear function of x which has the value a when $x=0$, and which is doubled when x increases from 1 to 4.

20. If y varies as x, then y is a linear function of x. Is the converse true?

21. Copy the following table and fill in the blanks, assuming that y is a linear function of x.

x	-3	1	4	......
y	-33	......	16	58

22. It is given that y is the sum of three quantities, of which the first is constant, the second varies as x, and the third varies inversely as x. Three sets of corresponding values of y and x are

$$y=9,\ x=1;\ y=8,\ x=2;\ y=9,\ x=3.$$

Find the relation between y and x.

23. Prove that, if y varies as x, then any two values of x have to each other the same ratio as the corresponding values of y. Prove also the converse.

24. The velocity with which water issues from an orifice varies as the square root of the depth of the orifice below the surface of the water. A vessel four feet deep is emptied by a tap at the bottom. If, when the vessel is full, the velocity of flow from the tap is 16 feet per second, find the velocity when the water is one foot deep.

25. The strength of a rectangular beam varies jointly as its width and the square of its depth. Of two beams having the same area of cross-section, the first is square and the width of the second is 4/9 of its depth. Compare the strengths of the two beams.

26. The volume of a sphere varies as the cube of its diameter. Find the diameter of a sphere equal in volume to the combined volumes of three spheres of diameters 3, 4, and 5 inches, respectively.

27. The volume of a circular cone varies jointly as its height and the square of the radius of its base. What is the effect on the volume of (i) doubling the radius of the base, (ii) doubling both radius and height, (iii) doubling the radius and halving the height? If the radius is trebled, by how much must the height be changed to leave the same volume?

28. The power necessary to propel a boat varies as the cube of the boat's speed relative to the water. If, in a river with a current of 4 miles an hour, 400 horse-power will propel a boat downstream at 20 miles an hour, what horse-power will drive the boat upstream at the same speed?

29. The time of the swing of a pendulum varies as the square root of its length. If a pendulum 39 inches long swings 299 times in 5 minutes, by how much should it be shortened to make it swing once a second?

30. The intensity of illumination of an object varies inversely as the square of the distance of the object from the source of light. A person 30 feet from a lamp finds the light too dim for reading. How much closer should he move to the lamp to double the amount of illumination on his paper?

C

31. Find the linear function which takes the values m and n when x equals c and d respectively. Show that this function may be put into the form

$$f(x)=\frac{x-d}{c-d}\,m+\frac{x-c}{d-c}\,n.$$

32. If y is the same linear function of x as x is of y, show that either $y=x$ or $x+y$ is a constant.

33. A linear function may be used for extrapolating in the same way as for interpolating. In extrapolating, one estimates the value of the function for a value of the independent variable outside the given range of values. Are the results of this process likely to be as reliable as those of interpolation? Given

$$f(3.6)=5.2874$$
$$f(3.7)=5.2658,$$

find, by linear extrapolation, $f(3.75)$.

34. If a pendulum 76.8 inches long makes 2570 oscillations per hour, find the length of a pendulum which makes 3 oscillations in 2 seconds. (Refer to ex. 29).

35. The force of attraction between two spherical masses varies directly as the product of the two masses and inversely as the square of the distance between their centres (Newton's law of gravitation). Compare the attraction of the sun on the earth with that on Jupiter, given that (*a*) the earth's distance from the sun is 93 million miles, (*b*) Jupiter's distance from the sun is $483\frac{1}{3}$ million miles, (*c*) Jupiter's mass is 317.7 times that of the earth.

36. Using the law of gravitation (see ex. 35), and the fact that a sphere attracts as though all its mass were at its centre, find what would be the weight, on the moon, of a man who weighs 160 pounds on the earth. (Radius of earth$=$3960 miles; radius of moon$=$1080 miles; density of earth$=$5.5, and that of moon$=$3.4, compared with water.)

37. A man who weighs 160.0 pounds at sea level is found to weigh 159.6 pounds at a height of 5 miles above sea level. Find the approximate radius of the earth. (Use the law of ex. 36.)

38. Kepler's third law of planetary motion states that the square of the time of a planet's revolution about the sun varies as the cube of its mean distance from the sun. Taking the earth's mean distance from the sun as 93 million miles and the length of a year on Venus as 225 days, find the mean distance of Venus from the sun.

39. The laws of vibrating strings state that, for a stretched string, the number of vibrations per second varies inversely as the length, directly as the square root of the tension, and inversely as the diameter. Two musical strings of the same material and the same diameter have lengths of 20 cm. and 50 cm. respectively. The first is stretched by a weight of 4 kgm. By what weight must the other be stretched in order that it may emit a note one octave lower than that of the first?

40. The resistance to sliding a stone on the ground varies as the weight of the stone and the weight varies as the cube of the diameter. The resistance which a stream is able to overcome in moving a stone varies jointly as the square of the diameter and the square of the velocity of the stream. Show that, if the velocity of the stream is doubled, it can move a stone 64 times as heavy as before.

CHAPTER III

QUADRATIC FUNCTIONS AND QUADRATIC EQUATIONS

11. Definition and examples. *The function* ax^2+bx+c, *where* a, b, *and* c *are constants and* $a \neq 0$, *is called a* ***quadratic function*** *of* x. Thus x^2, $1-x^2$, $(x+2)^2$, $3x^2-2x+5$, $(x-a)^2+b^2$, are examples of quadratic functions.

Some of the importance of the subject arises from the fact that quadratic functions occur in many formulas of geometry, physics, and other subjects. This is illustrated by the following examples:

(*a*) the area of a square of side x is x^2;

(*b*) the area of a circle of radius r is πr^2;

(*c*) the area of the surface of a sphere of radius r is $4\pi r^2$;

(*d*) if a stone is thrown upward with a velocity of 25 feet per second, its distance above the starting point after t seconds is $25t-16t^2$;

(*e*) a mass m, moving with velocity v, possesses energy of motion or kinetic energy, given by the formula $\frac{1}{2}mv^2$.

12. The graph of a quadratic function. Particular case. The fact that the graph of a linear function is a straight line puts into evidence in a vivid manner the way in which a linear function varies. In studying the variation of a quadratic function, we shall again make the graph our tool. As a first example of a quadratic function, we take x^2-4x-5. This function will be denoted by y, giving the equation $y=x^2-4x-5$. By assigning values to x and computing the corresponding values of y, we build up the following table:

x	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
y	27	16	7	0	-5	-8	-9	-8	-5	0	7	16	27

The points (x, y) corresponding to the values in this table may now be plotted and joined by a smooth curve to give the graph shown in fig. 12. For values of x greater than 4, the term x^2 is numerically the greatest of the three terms which make up the function, and it becomes increasingly dominant in magnitude as x increases. The same situation holds as x decreases algebraically below -4. Since the term x^2 is positive, this means that the ordinate of the curve increases indefinitely and the curve rises indefinitely both on the left and on the right. Another property of the curve which may be conjectured from the table of values or from the graph is that it is symmetrical about the line $x=2$. The geometric fact that this line is an axis of symmetry corresponds to the algebraic fact that, when $x=2+a$, y has the same value as when $x=2-a$, for any value of a. A general method of proving this latter result will be given in sec. 13. For the present, it may be proved by direct substitution. Thus, when $x=2-a$,

$$\begin{aligned} y &= (2-a)^2-4(2-a)-5 \\ &= 4-4a+a^2-8+4a-5 \\ &= a^2-9. \end{aligned}$$

This result is unaltered by changing the sign of a. Hence y has the same value when $x=2+a$. This proves the symmetry of the curve.

12.1. The behaviour of a quadratic function. In order to acquire the habit of interpreting graphs, the student should observe the manner in which changes in x bring about corresponding changes in the function. As x increases, starting from large negative values, the function, y, is positive and decreasing and takes the value 0 when $x=-1$. For values of x between -1 and 5, the function is negative, decreasing to its smallest value when $x=2$ and then increasing to take the value 0 again when

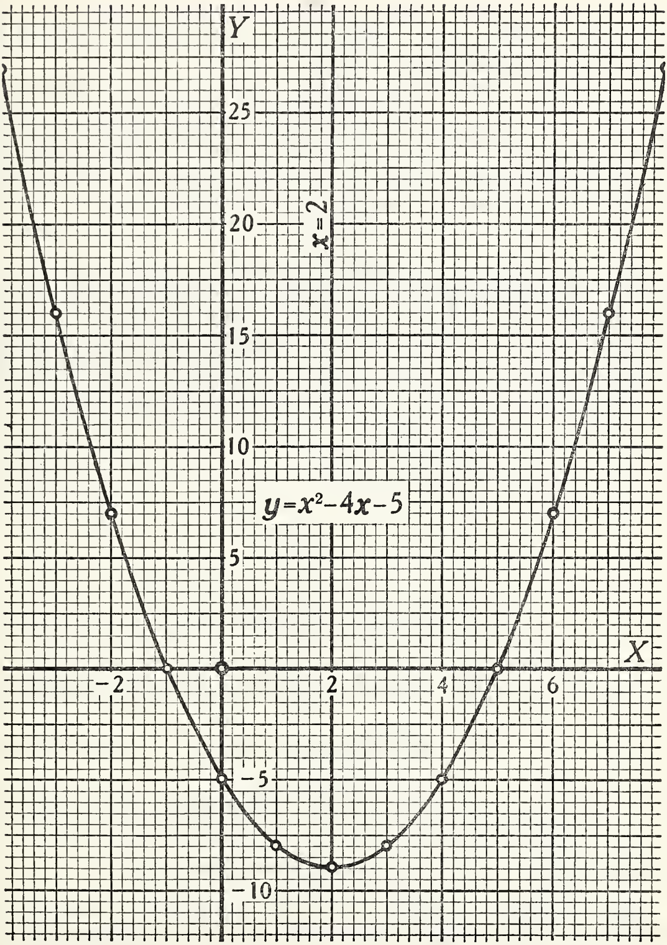


FIG. 12

$x=5$. As x increases beyond 5, the function is positive and increasing. The graph, unlike that of a linear function, contains no points for which $y<-9$.

The curve is called a *parabola*; its axis of symmetry, which is parallel to the axis of y, is called the *axis of the parabola*, and the point in which the axis meets the parabola is its *vertex*. In the example given, the vertex is seen from the graph to be the point on the curve at which the value of y is least.

13. The graph of a quadratic function. General case. A study of the general quadratic function ax^2+bx+c shows that its graph resembles that of the particular function of sec. 12. Denoting this general function by y, we make use of the following analysis which is of much help in the study of quadratics and will be referred to as the method of ***completing the square.***

$$\begin{aligned} y &= ax^2+bx+c \\ &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a} \\ \text{(1)} \qquad &= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}. \end{aligned}$$

This form shows that, when $x=\dfrac{-b}{2a}+k$, y has the same value as when $x=\dfrac{-b}{2a}-k$, for any value of k. Hence *the graph has the line* $x=\dfrac{-b}{2a}$ *as an axis of symmetry.* This axis cuts the curve at the point $\left(\dfrac{-b}{2a}, \dfrac{4ac-b^2}{4a}\right)$, which is called the vertex.

The quadratic function can be written in the form $x^2(a+b/x+c/x^2)$. When x is numerically large, whether positive or negative, b/x and c/x^2 are close to 0 so that the function is approximated by ax^2, which becomes the dominant term in the expression. Hence if $a>0$, the function is positive for values of x numerically large and increases rapidly as the numerical value of x increases. This fact is expressed by saying that the parabola opens upward. On the other hand, if $a<0$, the term which dominates the function for numerically large values of x is negative. The curve which is its graph is still a parabola of which the axis and vertex are found as in the previous case. However, the parabola in this case opens downward.

14. Determination of a quadratic function from three known values. It was pointed out in chapter II, page 29, that a linear function can be determined when its values are known for two values of the independent variable. We consider now the analogous problem of the determination of a quadratic function. Since the general expression ax^2+bx+c contains three arbitrary constants, it is to be expected that the function is completely determined when its values are known corresponding to three values of x. The following examples show how the coefficients are found.

Example 1. Determine the quadratic function which has the values 10, 5, 7, when $x=0, 1, 3$, respectively.

Denoting the function by ax^2+bx+c and giving to x the values 0, 1, 3, we get the equations:

$$\begin{aligned} c &= 10 \\ a+\ b+c &= 5 \\ 9a+3b+c &= 7. \end{aligned}$$

When 10 is substituted for c, the last two equations are

$$\begin{aligned} a+\ b &= -5 \\ 9a+3b &= -3, \end{aligned}$$

whence $a=2$ and $b=-7$. The required function is therefore $2x^2-7x+10$.

Example 2. Find, if possible, the quadratic function which takes the values 3, 5, 7, when $x=1$, 2, 3, respectively.

By substitution in ax^2+bx+c, we get

$$\begin{aligned} a+b+c &= 3 \\ 4a+2b+c &= 5 \\ 9a+3b+c &= 7, \end{aligned}$$

whence

$$\begin{aligned} 3a+b &= 2 \\ 5a+b &= 2, \end{aligned}$$

which give $a=0$, $b=2$, and hence $c=1$. The function is $0x^2+2x+1$, which is not quadratic. The explanation is that the three points (1, 3), (2, 5), (3, 7) lie on the straight line $y=2x+1$ and not on a parabola.

The fact that a linear function of x is determined when its values are known for two values of x, corresponds to the fact that one and only one straight line can be found to pass through two given points. The fact that a function ax^2+bx+c is determined when its values are known for three values of x, may also be interpreted geometrically. If a is found to be 0, the three points lie on the straight line $y=bx+c$. If, however, the three points do not lie on a straight line, then $a\neq 0$; in this case the points lie on and determine a parabola with a vertical axis. Hence, one and only one such parabola can be found to pass through three points which are not in a straight line.

EXERCISES

1. Which of the following are quadratic functions and which are not? Give your reason for each answer.

(*a*) $3x^2+7x$, (*b*) $3x^2-7x$, (*c*) $3x^2$,

(*d*) $\dfrac{1}{2x^2+3x+5}$, (*e*) a^x+bx+c, (*f*) $3x^3-5x+2$, (*g*) $\dfrac{x^2+7}{x^2-7}$, (*h*) $\dfrac{x^2-4}{x+2}$, (*i*) $\dfrac{x^3+1}{x+1}$.

2. From the graph of fig. 12, read the answers to the following: What values of x make the function x^2-4x-5 equal to -5? For what values of x does this function vanish? For what values of x is it negative?

3. Tabulate the values of the function $-x^2+3x+4$ for integral values of x from -6 to 9 inclusive. Use this table (i) to describe the behaviour of the function as x increases from -6 to 9; (ii) to determine the axis of symmetry and so find the greatest value of the function; (iii) to find the roots of the equation $4+3x-x^2=6$.

4. Use the graph of fig. 13 to locate the axis of symmetry and to describe the behaviour of the function x^2-8x+7 as x increases from 0 to 8.

5. Tabulate the values of x^2-6x for integral values of x from -2 to 8 inclusive, and construct the graph of the function. What is its axis of symmetry? What is the smallest value of the function? In what interval does an increase in x produce an increase in the function?

6. What term must be added to x^2+ax in order to complete the square? State this rule in words.

7. Complete the squares of (*a*) x^2+6x, (*b*) x^2-10x, (*c*) $2x^2+5x$, (*d*) $3x^2-8x$.

8. For the function $3x^2-4x+5$, complete the square and find the axis of symmetry of the graph. Does the graph open upward or downward?

9. For the function $9-14x+8x^2$ complete the square and find the axis of symmetry of the graph. In which direction does the curve open?

10. Complete the square for the function $px^2+2qx+r$. State the conditions under which the graph will open downward and the axis of symmetry will lie on the positive side of the y-axis.

11. Under what condition is the graph of ax^2+bx+c symmetrical about the y-axis?

12. Draw the graph of $4x-x^2$. Hence sketch the graphs of $4x-x^2+4$ and $4x-x^2-4$.

13. Find the quadratic function of r which vanishes when $r=-3$ and which has the values -12 and 21 when r equals 1 and 4 respectively.

14. Find the quadratic function which takes the values -4, 2, -46, when $x=-1$, 2, 5, respectively.

15. Find the quadratic function which takes the values 0, 90, 9900, when $x=0$, 10, 100, respectively.

15. The quadratic equation. Graphical treatment. If in the relation $y=x^2-4x-5$ we set $y=0$, there results the *quadratic equation* $x^2-4x-5=0$. Since $y=0$, the values of x for which this equation holds (in other words, the *roots* of the equation) are the abscissas, -1 and 5, of the points where the graph (fig. 12) crosses the axis of x. For the equation

$$ax^2+bx+c=0, \qquad (a\neq 0) \tag{2}$$

a similar statement holds. In the relation of the parabola $y=ax^2+bx+c$ to the axis of x, there are three possibilities: (*a*) the curve may cut the axis of x in two distinct points, (*b*) the curve may touch the axis of x at the vertex of the parabola,

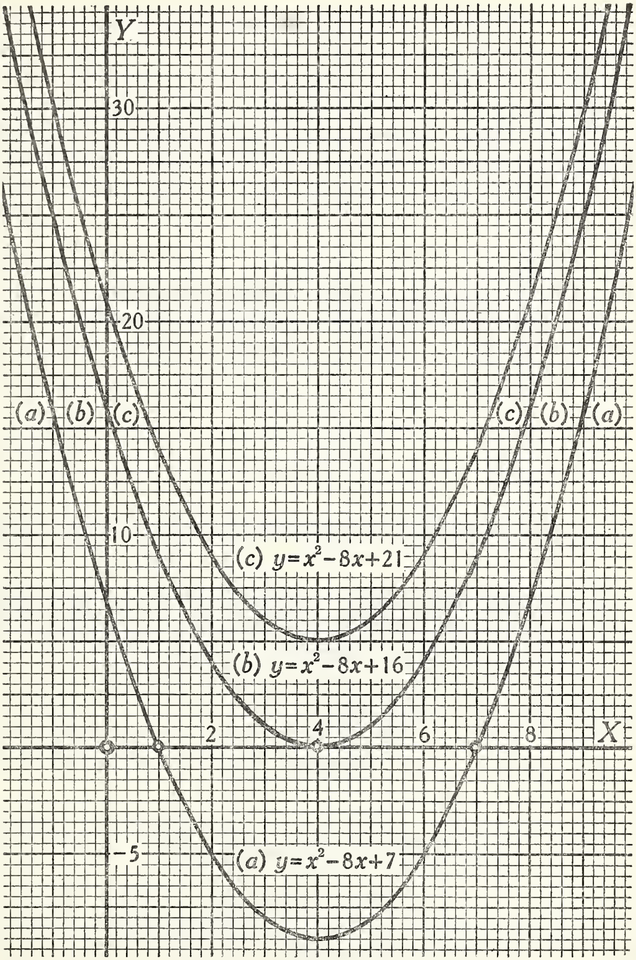


FIG. 13

(*c*) the curve may have no point in common with the axis of x, that is, it may lie wholly above or wholly below. Fig. 13 illustrates these possibilities. In case (*a*), the equation (2) has two roots which are real and different. In case (*b*), the equation has only one root. Since, however, this may be regarded as a limiting case in which two distinct roots have come together, it is useful to regard the equation in case (*b*) as having two equal roots. In case (*c*), equation (2) has no real roots.

16. The quadratic equation. Algebraic treatment. When a quadratic function can be easily expressed as the product of two linear factors, this provides the simplest way of solving the corresponding equation. Thus, the equation

$$6x^2-17x-14=0,$$

can be written

$$(2x-7)(3x+2)=0,$$

and, since this equation is satisfied if either factor is zero, we get $x=7/2$ and $x=-2/3$ as the roots of the equation.

For any quadratic equation the roots, when any roots exist, can be found by the method of completing the square. The work can be carried out as follows:

The equation

$$ax^2+bx+c=0$$

can be written

$$x^2+\frac{bx}{a}=\frac{-c}{a}.$$

Hence

$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

or

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}. \tag{3}$$

The left member of (3) is positive or zero for all real values of x. In the right member, the denominator $4a^2$ is necessarily positive. Whether or not the equation can be satisfied depends therefore on the numerator b^2-4ac. This expression, involving the three coefficients of the quadratic function ax^2+bx+c, is called the *discriminant* of the function or of the equation (2). From (3), we get the following information regarding the roots of (2):

(*a*) *If* $b^2-4ac>0$, *then* (2) *has two distinct roots, namely*

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}. \tag{4}$$

This is case (*a*) of sec. 15.

(*b*) *If* $b^2-4ac=0$, *then* (2) *has two equal roots, each of the value* $\frac{-b}{2a}$. This is case (*b*) of sec. 15.

(*c*) *If* $b^2-4ac<0$, *then* (2) *has no real roots.* This is case (*c*) of sec. 15.

It is to be noted that if the roots of a quadratic equation are known and also the sign of the coefficient of the square term, then a sketch can be made showing the character of the graph of the corresponding quadratic function. For the roots provide two points on the axis of x through which the graph passes, and the sign of the coefficient of the square term tells whether the graph opens upward or downward.

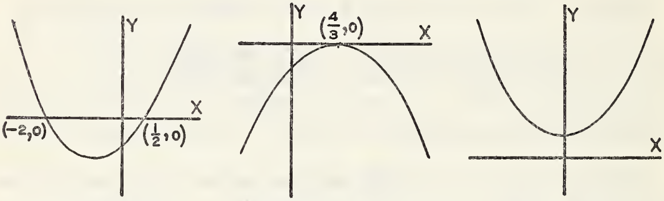
EXAMPLE 1. Sketch the graphs of the following functions: (*a*) $(2x-1)(x+2)$; (*b*) $-9x^2+24x-16$; (*c*) x^2+1.

We have the following information:
(*a*) $a>0$, and the roots of $(2x-1)(x+2)=0$ are $1/2$ and -2. Hence, the graph opens upwards and cuts the x-axis at $(1/2, 0)$ and $(-2, 0)$.

(*b*) $a<0$, and the roots of $-9x^2+24x-16=0$, or $-(3x-4)^2=0$, are both 4/3. Hence the graph opens downwards and touches the x-axis at (4/3, 0).

(*c*) $a>0$, and the equation $x^2+1=0$ has no real roots. Hence the graph opens upwards and has no point in common with the x-axis.

The sketches are therefore as shown in the figure.



(a) $(2x-1)(x+2)$ (b) $-9x^2+24x-16$ (c) x^2+1

EXAMPLE 2. By completing the square, solve $4x^2+28x+37=0$.

We have $$x^2+7x=-\frac{37}{4}.$$

Hence $$x^2+7x+\frac{49}{4}=\frac{49}{4}-\frac{37}{4}=\frac{12}{4}=3,$$

or $$\left(x+\frac{7}{2}\right)^2=3,$$

and $$x+\frac{7}{2}=\pm\sqrt{3},$$

whence $$x=-\frac{7}{2}\pm\sqrt{3}.$$

EXAMPLE 3. What information is furnished by the discriminant concerning each of the following equations and the corresponding quadratic functions?

(i) $2x^2-7x+4=0$; (ii) $-9x^2+30x-25=0$; (iii) $9x^2+8x+2=0$.

(i) The discriminant $= b^2-4ac$

$$= (-7)^2-4\cdot 2\cdot 4=49-32=17.$$

Hence the roots of the equation are real and unequal. The graph of $2x^2-7x+4$ cuts the x-axis in two distinct points.

(ii) The discriminant $= 30^2-4(-9)(-25) = 0$. Hence the roots of the equation are real and equal. The graph of $-9x^2+30x-25$ touches the x-axis.

(iii) The discriminant $= 8^2-4\cdot 9\cdot 2=64-72=-8$. Hence the equation has no real roots. The graph of $9x^2+8x+2$ has no point in common with the x-axis.

EXERCISES

1. Taking (2) as the type of a quadratic equation, read the values of a, b, and c in each of the following: $15x^2-37x=8$; $(x-3)^2=2x-5$; $(3x+2)^2=(4-x)^2$; $p(x+m)^2+q(x+m)+r=0$.

2. By factoring, solve

(*a*) $10x^2+7x-12=0$, (*b*) $7x^2+40x-12=0$,
(*c*) $14x^2+29x-15=0$, (*d*) $m^2+0.9m-0.36=0$.

3. By completing the squares, solve (*a*) $x^2+2x-2=0$, (*b*) $5x^2+12x+6=0$, (*c*) $1+5x-5x^2=0$, (*d*) $25r^2-60r-72=0$.

4. By using the formula (4), solve (*a*) $2x^2-2\sqrt{6}x+3=0$, (*b*) $2x^2-7x+4=0$, (*c*) $px^2+2qx+r=0$, (*d*) $t^2-at-a^2=0$.

5. Use the graph of fig. 12 to find the roots of

(*a*) $x^2-4x-5=-8$, (*b*) $x^2-4x-5=7$.

6. Use the graphs of fig. 13 to find the real roots, if any, of (*a*) $x^2-8x+7=0$, (*b*) $x^2-8x+7=-5$, (*c*) $x^2-8x+16=-5$, (*d*) $x^2-8x+21=5$. What is the smallest value that the function x^2-8x+7 can assume?

7. Without constructing tables of values, make sketches of the graphs of the following functions:

(*a*) x^2-4; (*d*) $-x^2-4$; (*g*) $(2x+5)(x-3)$;
(*b*) $4-x^2$; (*e*) $4x-x^2$; (*h*) $-(x-2)(x+3)$;
(*c*) x^2+4; (*f*) x^2+6x+9; (*i*) $2x^2+7x-15$.

8. Which of the following equations have real roots and which have not?
(*a*) $30x^2+25x+4=0$, (*b*) $49x^2-28x+4=0$, (*c*) $x^2+mx+m^2=0$.

9. What information does the discriminant give concerning the graph of each of the following functions?
(*a*)x^2+x, (*b*) $7x^2-8x+3$, (*c*) $9x^2-6\sqrt{2}mx+2m^2$, (*d*) $3x^2-7x+2$.

17. The system of real numbers. It will be profitable at this point to renew our inquiry regarding the system of real numbers which are, so far, the only numbers of which we have any knowledge. Historically, the first numbers to be used were the positive integers 1, 2, 3, · · · . For the purpose of commerce and engineering, the integers were found insufficient and the system of numbers was extended by the introduction of fractions. More difficulty was encountered with the concept of negative numbers, but they finally gained admission to the family of numbers and have proved their usefulness. The number 0 is so familiar to us that we generally fail to realize the great advance which its introduction marked in mathematics. The numbers so far mentioned, positive and negative integers and fractions and the number zero, make up the system of *rational* numbers. This fact is made explicit in the following definition.

DEFINITION. *A rational number is one which may be represented in the form* $\frac{a}{b}$, *where a and b are integers and* $b \neq 0$.

In spite of the great utility of rational numbers, they have been found to be insufficient for even some simple problems.

and the study of this question has led to some interesting and curious properties of numbers. Rational numbers may be found as close in value as we please to a given rational number. If rational numbers are represented by points on a line, as explained in sec. 1, we may call the corresponding points rational points; thus we may speak of the point 3, the point —4, the point $\frac{5}{2}$, etc. Between any two rational points, we may find a third since, for example, the average of the two corresponding numbers is a rational number which lies in value between them. Thus, between $\frac{1}{2}$ and $\frac{1}{3}$ lies the point $\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{5}{12}$; between $\frac{1}{2}$ and $\frac{5}{12}$ lies the point $\frac{1}{2}\left(\frac{1}{2}+\frac{5}{12}\right)$, etc. This process of locating more and more rational points on a segment of a straight line has no end.

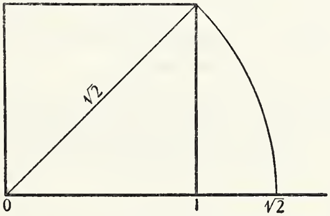


FIG. 14

The rational points on a straight line are therefore said to be *everywhere dense* and the rational numbers are said to form an everywhere dense set.

Since the set of rational numbers is everywhere dense, it might be supposed that all real numbers are rational, in other words that all points on our straight line represent rational numbers. That this is not true, however, is seen from the following simple example. If the side of a square is 1 unit (fig. 14) and the diagonal x units, we find $x^2=2$. We therefore introduce the symbol $\sqrt{2}$ for the length of this diagonal, relying on the geometric evidence for the existence of a number corresponding to this symbol. Is this number rational? The following proof shows that it is not.

Suppose that it is possible to represent $\sqrt{2}$ by a rational number $\frac{a}{b}$, and that the fraction $\frac{a}{b}$ is in its lowest terms. Then, since $\sqrt{2}=\frac{a}{b}$, it follows that $2=\frac{a^2}{b^2}$ and $a^2=2b^2$. Hence a^2 is an even number. Therefore a is an even number, since the square of an odd number is odd. Now $b^2=\frac{a^2}{2}=\frac{a}{2}\cdot a$ and, since $\frac{a}{2}$ is an integer, it follows that $\frac{a}{2}\cdot a$ is an even number and hence b^2 is an even number. Finally, it follows that b is an even number. If a and b are both even then $\frac{a}{b}$ is not in its lowest terms. This contradiction proves the impossibility of representing $\sqrt{2}$ by a rational number.

This example shows the necessity for other numbers besides rational numbers.* The reader is already familiar with the use of such irrational numbers as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, π, and others. From a geometrical point of view, all points on the axis of x which do not represent rational numbers will be said to represent *irrational numbers. The rational and the irrational numbers together make up the set of real numbers.*

18. Imaginary numbers. It has been seen in sections 15 and 16 that there are quadratic equations which have no real roots.

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*The recognition of the need for irrational numbers was one of the achievements of the Pythagorean school of mathematics which, under Pythagoras, flourished in Croton, Italy, around 520 B.C.

This fact appears from both geometric and algebraic considerations. On the geometric side, the non-existence of real roots is shown by the fact that the parabola in question has no points in common with the axis of x (fig. 13c). On the algebraic side, the proof that no real roots exist when $b^2-4ac<0$ is provided by equation (3), since a number which is positive or zero cannot be equal to a negative number.

One of the simplest equations with no real roots is $x^2+1=0$. With regard to this equation, the situation may be compared with that of the equation $x+1=0$ before negative numbers were introduced. As long as the only numbers in use were positive numbers, it could be correctly maintained that $x+1=0$ has no solution. In the same way, as long as all the numbers used were real numbers it was correctly maintained that there are no numbers which satisfy $x^2+1=0$; when some mathematicians insisted on introducing such "numbers", these were termed imaginary. With the definitions which are now accepted for these new numbers, it is clear that the word imaginary does not describe them any more than it does, for example, the number -1. Nevertheless the term ***imaginary*** has persisted as the technical name of the new numbers and is an interesting reminder of the difficulty which they had in forcing their way into the system of numbers.*

For a systematic account of imaginary numbers the reader who is curious is referred to appendix II, page 314. At this point we give only a statement of results and examples of the use of the numbers.

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*With regard to the futility of the question "Are the imaginary numbers imaginary and are they numbers?", Professor A. N. Whitehead makes the following delightful comment: "It cannot be too clearly understood that, in science, technical terms are arbitrarily assigned, like Christian names to children. There can be no question of the names being right or wrong. They may be judicious or injudicious; for they can sometimes be so arranged as to be easy to remember, or so as to suggest relevant and important ideas. But the essential principle involved was quite clearly enunciated in Wonderland to Alice by Humpty Dumpty, when he told her, apropos of his use of words, 'I pay them extra and make them mean what I like.'"

Introduction to Mathematics.

The important fact to recognize at this stage is that we are concerned with an extension of the number system so as to provide roots for the equation $x^2+1=0$ and for similar equations, just as the introduction of negative numbers provided a root for $x+1=0$. We shall aim to preserve the rules of operation which apply to the addition, subtraction, multiplication, and division of real numbers. Thus, if a meaning is to be given to the equation $x^2+1=0$, we shall agree that that $x^2=-1$ shall have the same meaning. We now introduce (by means of the definitions of appendix II) two numbers of a new kind, denoted by $\sqrt{-1}$ and $-\sqrt{-1}$, in order to satisfy this equation. As an alternative symbol for $\sqrt{-1}$, we use the letter i. Since i and $-i$ satisfy the equation, it follows that $i^2=-1$ and that $(-i)^2=-1$.

The symbol $a+b\sqrt{-1}$ or $a+bi$, where a and b are real numbers, is called a ***complex number.*** If $b=0$, this complex number is the real number a, and if $a=0$, it is the ***pure imaginary number*** bi. The set of all real numbers and likewise the set of all pure imaginary numbers are now seen to be included in the more comprehensive set of all complex numbers.

The following illustrations show how the principal algebraic operations are performed with complex numbers.

(i) $x+yi=0$ is true if, and only if, $x=0$ and $y=0$.

Thus, an equation such as $2-3i=0$ is impossible.

$x+yi=3+4i$ is true if, and only if, $x=3$ and $y=4$.

(ii) (*Addition*)

$$(3+5i)+(6-7i)=3+6+(5-7)i=9-2i.$$

(iii) (*Subtraction*)

$$(7+4i)-(3+7i)=7-3+(4-7)i=4-3i.$$

(iv) (*Multiplication*)

$$(3+5)(7+2i)=21-10+(6+35)i=11+41i.$$

The rule for multiplication can be written

$$(a+bi)(x+yi)=ax-by+(ay+bx)i.$$

As an important special case, we have

$$(a+bi)(a-bi)=a^2+b^2.$$

The numbers $a+bi$ and $a-bi$ are called ***conjugate*** complex numbers.

(v) (*Division*)

$$\frac{18+25i}{3+2i}=\frac{18+25i}{3+2i}\times\frac{3-2i}{3-2i}=\frac{54+50+(75-36)i}{9+4}$$

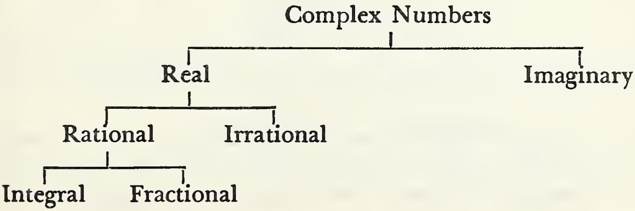
$$=\frac{104+39i}{13}=8+3i.$$

(vi) $\sqrt{-16}=4\sqrt{-1}=4i$. Similarly, $\sqrt{-7}=i\sqrt{7}$.

From the foregoing illustrations, it is clear that complex numbers may be freely employed in formal operations. ***The symbol i or $\sqrt{-1}$ is used as if it were a real number, with the additional rule that $i^2=-1$.*** It is to be noted, however, that, in the drawing of graphs, the coordinates of all points are real. Imaginary numbers cannot be coordinates of points in the usual graphical representation.

EXERCISES

1. Below is a tabular classification of numbers and, following this table, a list of numbers. State the precise class to which each number in the list belongs.



$$0, \qquad 7\tfrac{3}{4}, \qquad \frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}} \qquad \sqrt{20}, \qquad \sqrt{\frac{4}{9}}, \qquad \sqrt{-100},$$

$$(\sqrt{2}-1)^2, \qquad \frac{1}{\sqrt{10}-3}, \qquad (\sqrt{-6})^2, \qquad 2+\sqrt{-3}.$$

2. If m is an integer and $i=\sqrt{-1}$, find the values of $(i)^{4m}$, $(i)^{4m+1}$, $(i)^{4m+2}$, and $(i)^{4m+3}$.

3. Write the following complex numbers in the form $a+bi$, where a and b are real:

$$\frac{2}{3i}, \qquad (4-3i)^2, \qquad (2+i)^3, \qquad \frac{1}{1+i}, \quad \frac{2+i}{2-i}.$$

4. Express each of the following products in the form $a+bi$:

(*a*) $(2-3i)(2+3i)$, (*b*) $(\sqrt{3}+\sqrt{-8})(\sqrt{3}+\sqrt{-2})$,

(*c*) $(1+i)(1+2i)(1+3i)$.

5. Express each of the following quotients in the form $a+bi$:

(*a*) $\dfrac{6-3i}{i}$, (*b*) $\dfrac{50i}{7+i}$, (*c*) $\dfrac{12+5i}{3-2i}$.

19. The number and character of the roots of $ax^2+bx+c=0$. As a result of the use of imaginary numbers, the equation $x^2+1=0$ has the roots i and $-i$. Returning now to the general quadratic, we write it in the form (3), page 58.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

If $b^2-4ac<0$, this equation has no real roots. However,

$$\pm\frac{\sqrt{b^2-4ac}}{2a}, \text{ or } \pm i\frac{\sqrt{4ac-b^2}}{2a},$$

are two imaginary numbers which satisfy the equation when substituted for $x+\frac{b}{2a}$. Hence two imaginary roots of the equation are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{4}$$

This is also the formula (page 59) which gives the real roots of the equation when $b^2-4ac\geqq 0$.

It follows that a quadratic equation has two roots which may be real and different, real and equal, or imaginary. There remains to be answered the question of whether more than two roots are ever possible. It will now be shown by an indirect proof that no quadratic equation can have as many as three distinct roots.

Suppose, if possible, that x_1, x_2, x_3 are three distinct roots of $ax^2+bx+c=0$. Then, since these numbers satisfy the equation,

$$\begin{aligned} ax_1^2+bx_1+c &= 0, \\ ax_2^2+bx_2+c &= 0, \\ ax_3^2+bx_3+c &= 0. \end{aligned}$$

By subtraction, these equations yield

$$\begin{aligned} a(x_1^2-x_2^2)+b(x_1-x_2) &= 0, \\ a(x_2^2-x_3^2)+b(x_2-x_3) &= 0. \end{aligned}$$

Since, by hypothesis, x_1-x_2 and x_2-x_3 are different from zero, these factors can be removed from the left members of the two equations respectively. Hence,

$$\begin{aligned} a(x_1+x_2)+b &= 0, \\ a(x_2+x_3)+b &= 0. \end{aligned}$$

Again, by subtraction,

$$a(x_1-x_3)=0.$$

Now $a\neq 0$ since the original equation is quadratic, and $x_1-x_3\neq 0$ by hypothesis. Hence this last equation is a contradiction and the assumption that three distinct roots exist is incorrect.

The result may now be summed up as follows. The equation $ax^2+bx+c=0$, in which $a\neq 0$, has two and only two roots, which are given by the formula (4). The character of these roots is determined by the discriminant b^2-4ac.

The roots of $ax^2+bx+c=0$ are
(i) real and different if $b^2-4ac>0$,
(ii) real and equal if $b^2-4ac=0$,
(iii) imaginary if $b^2-4ac<0$.

Example 1. Determine the character of the roots of

(*a*) $4x^2-4\sqrt{3}x+3=0$, (*b*) $3x^2-x+5=0$,

(*c*) $x^2+(p+1)x+p-1=0$, where p is a real number.

(*a*) The discriminant $=48-4\cdot 4\cdot 3=0$. Hence the roots are real and equal.

(*b*) The discriminant $=1-4\cdot 3\cdot 5=-59$. Hence the roots are imaginary.

(*c*) The discriminant
$$=(p+1)^2-4(p-1)$$
$$=(p-1)^2+4,$$
which is greater than zero. Hence the roots are real and different. (Note that this conclusion follows only if p is a real number. If, for example, $p=1+3i$, a complex number, then the discriminant $(3i)^2+4=-9+4=-5$.)

Example 2. For what integral values of k does the equation $x^2-kx+4=0$ have imaginary roots?

The discriminant $=k^2-16$. The roots are imaginary if $k^2<16$, and the integral values of k which satisfy this inequality are 0, ± 1, ± 2, ± 3.

20. Relations between the roots and coefficients of $ax^2+bx+c=0$.

If the roots given in (4) are denoted by m and n so that

$$m=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a},\quad n=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a},$$

we find, by addition and multiplication, respectively,

$$\boxed{m+n=-\frac{b}{a},\quad mn=\frac{c}{a}.} \tag{5}$$

By means of these formulas, the sum and product of the roots of a quadratic equation can be found by inspection.

The equation $ax^2+bx+c=0$

can be written $x^2+\frac{b}{a}x+\frac{c}{a}=0,$

or, in terms of the roots m and n,

$$\boxed{x^2-(m+n)x+mn=0.}$$

The expressions $m+n$ and mn are symmetric functions of m and n in the sense that neither function is altered when m and n are interchanged. Other examples of symmetric functions of m and n are

$$m^3+n^3,\ m^2-3mn+n^2,\ \left(\frac{m+n}{mn}\right)^3,\ \text{etc.}$$

These functions can be expressed in terms of $m+n$ and mn, which are called the *elementary symmetric functions* of m and n. The following example illustrates the method.

EXAMPLE 1. If m and n are the roots of $x^2-3x+5=0$, find the value of m^3+n^3.

We have $m+n=3,\ mn=5.$

Also
$$\begin{aligned} m^3+n^3 &= (m+n)^3-3m^2n-3mn^2 \\ &= (m+n)^3-3mn\,(m+n) \\ &= 3^3-3\cdot5\cdot3 = 27-45 = -18. \end{aligned}$$

By means of such reductions of symmetric functions, equations can be formed, the sum and product of whose roots are given symmetric functions of the roots of a given equation.

EXAMPLE 2. Form the equation whose roots are the squares of those of $2x^2-6x+3=0$.

The relations (5) make it unnecessary to solve the given equation. Denoting its roots by m and n, we have

$$m+n=3,$$
$$mn=\frac{3}{2}.$$

Since the roots of the required equation are m^2 and n^2, it is

$$x^2-(m^2+n^2)x+m^2n^2=0.$$

Now
$$\begin{aligned} m^2+n^2 &= (m+n)^2-2mn \\ &= 3^2-2\cdot\frac{3}{2}=9-3=6, \end{aligned}$$

and
$$m^2n^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}.$$

The required equation is therefore

$$x^2-6x+\frac{9}{4}=0$$

or
$$4x^2-24x+9=0.$$

20.1. Factors of ax^2+bx+c. Making use of relations (5), we have

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a\{x^2-(m+n)x+mn\}. \\ &= a(x-m)(x-n) \end{aligned}$$

where m and n are the roots of the corresponding equation.

The use of imaginary numbers makes it possible to express the quadratic function as the product of two linear factors no matter what real values a, b, and c may have. The factorization into real factors is possible only when $b^2-4ac\geqq 0$.

EXAMPLE 3. Express $3x^2+10x+19$ as the product of two linear factors.

The roots of $3x^2+10x+19=0$ are

$$\frac{-10\pm\sqrt{100-12\cdot 19}}{6}=\frac{-5\pm\sqrt{25-57}}{3}$$
$$=\frac{-5\pm\sqrt{-32}}{3}.$$

Hence the required factors are

$$3\left(x+\frac{5-4\sqrt{-2}}{3}\right)\left(x+\frac{5+4\sqrt{-2}}{3}\right)$$

or $$\frac{1}{3}(3x+5-4\sqrt{-2})(3x+5+4\sqrt{-2}).$$

The student has now had some experience in studying *functions* and *equations.* A sharp distinction should be made between these two ideas. Thus $x+1$ is a linear function in which x may be any number whatever; on the other hand, $x+1=0$ is a linear equation which is satisfied by only one value of x, namely -1. In order to emphasize the distinction between a quadratic function and a quadratic equation, the salient facts are recorded in the following table.

The quadratic function ax^2+bx+c	The quadratic equation $ax^2+bx+c=0$
a, b, and c are constants and $a \neq 0$.	
x is any number.	x is a number which gives the function the value 0.
For real values of x the function has a parabolic graph.	If the roots of the equation are real, they are given by the intersections of the x-axis with the graph of the function.
The function is the product of two linear factors, $a(x-m)(x-n)$, where m and n are the roots of the corresponding equation.	The equation has two roots $m=(-b+\sqrt{b^2-4ac})/2a$ $n=(-b-\sqrt{b^2-4ac})/2a$

EXERCISES

1. Determine the character of the roots of:
(*a*) $x^2+5x+3=0$, (*b*) $2x^2-7x+8=0$, (*c*) $4x^2-20x+25=0$.

2. Write the quadratic equation whose roots are:
(*a*) -8 and 2, (*b*) $7/2$ and $-2/3$, (*c*) $2m$ and $3n$, (*d*) $a+b$ and $a-b$.

3. Write the quadratic equation for which the sum and product of the roots are, respectively: (*a*) -4, 8, (*b*) 10, 21, (*c*) $11/2$, 15, (*d*) $3b$, $2b^2$.

4. Complete the following equations: (*a*) $x^2-5x+\dots=0$, given that one root is 3; (*b*) $x^2+\dots-12=0$, given that one root is 6; (*c*) $x^2-ax+\dots=0$, given that one root is b; (*d*) $x^2+\dots+3c^2=0$, given that one root is c.

5. (*a*) Find the roots of (i) $x^2+4x+1=0$, (ii) $4h^2-4h+5=0$. (*b*) Express x^2+4x+1 and $4h^2-4h+5$ as the products of linear factors.

6. Express as the product of linear factors:
(*a*) $3x^2+7$, (*b*) x^2-4x-1, (*c*) $-x^2+x-1$, (*d*) $4x^2-17x+15$.

7. Form the equation whose roots are (i) the reciprocals, (ii) the squares of the reciprocals, of those of $5x^2-8x+6=0$.

8. If m and n are the roots of $x^2+px+q=0$, evaluate in terms of p and q:

(*a*) m^2+mn+n^2, (*b*) $\dfrac{1}{3m-2n}+\dfrac{1}{3n-2m}$, (*c*) $\dfrac{m}{n}+\dfrac{n}{m}$.

9. If m and n are the roots of $px^2+qx+r=0$, form the equation whose roots are

(*a*) $m+5$ and $n+5$, (*b*) m^3 and n^3, (*c*) $m-n$ and $n-m$.

10. If a and c have unlike signs, what is the character of the roots of $ax^2+bx+c=0$?

11. By using the factored form of ax^2+bx+c, show that a quadratic equation cannot have more than two roots.

21. The sign of ax^2+bx+c. For real values of x, a quadratic function with real coefficients takes only real values. In solving the quadratic equation, we have found the condition on the coefficients in order that the corresponding function should have the value zero for real values of x. This condition is $b^2-4ac \geqq 0$. We shall now examine the conditions under which the function is positive and those under which it is negative, for real values of x.

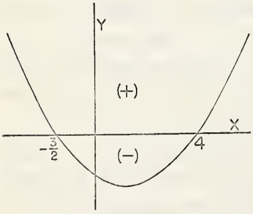
We have seen (sec. 13) that the graph of ax^2+bx+c, $(a \neq 0)$ is always a parabola with axis parallel to the axis of y, and that the parabola opens upward if $a>0$ and downward if $a<0$. If $a>0$, the graph may lie wholly above the axis of x (fig. 13c), it may touch the axis of x (fig. 13b), or it may cut the axis of x (fig. 13a); it cannot lie wholly below the axis of x. The points where the graph cuts the axis of x (cf. fig. 13a) are the points whose abscissas are the roots of $ax^2+bx+c=0$. The only case in which the value of the function is negative arises when this equation has real roots and the value of x lies between these roots.

Consider now the case $a<0$. Since the parabola now opens downward, it may lie wholly below the axis of x (fig. 15c), it may touch the axis of x (fig. 15b), or it may cut the axis (fig. 15a). The only case in which the value of the function is positive is that in which the equation has real roots and x lies between these roots.

Our conclusion is that, *for real values of the letters involved, the predominating sign of the function* ax^2+bx+c *is the sign of* a. *When the roots of* $ax^2+bx+c=0$ *are imaginary, the sign is always that of* a. *The function has values opposite in sign from that of* a *only when the equation has real, unequal roots and the value of* x *lies between these roots.*

In order to determine the sign of a quadratic function the student is advised to sketch the graph. For this purpose all that is necessary is the nature of the roots, their values if real, and the sign of the coefficient of the square term.

EXAMPLE 1. For what values of x is the function $2x^2-5x-12$ negative?



Since $2x^2-5x-12=(2x+3)(x-4)$, the roots of the corresponding equation are $-3/2$ and 4. Since the coefficient of x^2 is positive, the graph can be sketched as shown. It follows that the function is negative only when x lies between $-3/2$ and 4. This conclusion can also be drawn from a consideration of the factors $(2x+3)(x-4)$. If $x<-3/2$, then both factors are negative; if $x>4$, both factors are positive; if x lies between $-3/2$ and 4, then $2x+3$ is positive and $x-4$ is negative.

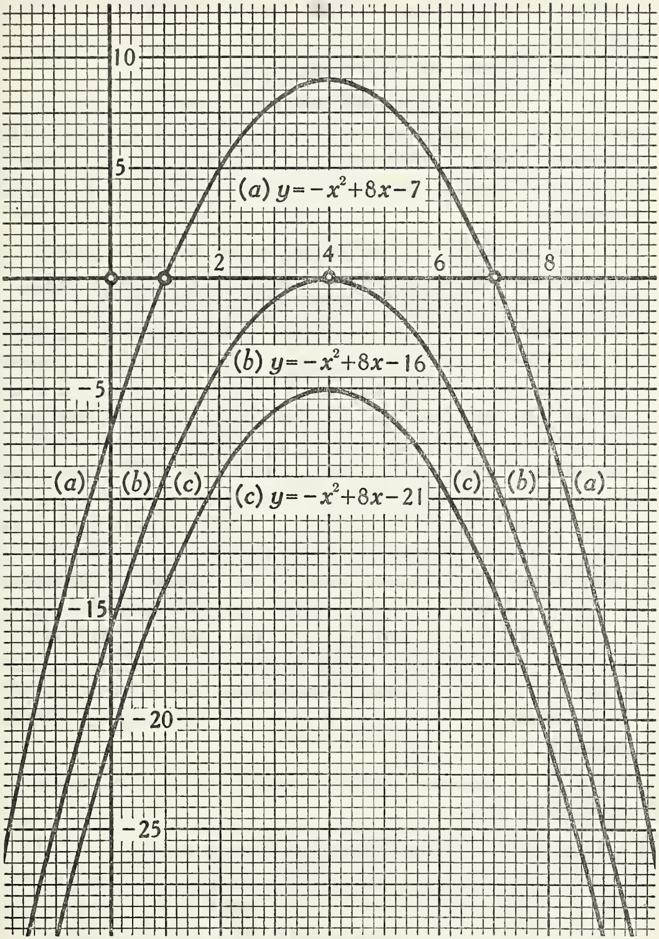


FIG. 15

EXAMPLE 2. Show that $-7+2z-3z^2$ is negative for all real values of z.

The discriminant of this quadratic is

$$2^2-4(-3)(-7)=4-84=-80.$$

Since this is negative, the corresponding equation has imaginary roots. Hence the sign of the function is, for all real values of z, the same as that of the coefficient of z^2, which is negative.

22. The maximum or minimum value of ax^2+bx+c. In sections 22 and 23, all numbers considered, whether values of coefficients or of variables, will be assumed to be real.

One significant respect in which the graphs of quadratic functions differ from those of linear functions is that, whereas a linear function can take any value whatever, the values which a quadratic function can take are restricted. A glance at figures 13 and 15 will remind us that, if $a>0$, then ax^2+bx+c has a minimum value but no maximum, and if $a<0$, it has a maximum but no minimum. The minimum or maximum is the ordinate of the vertex of the parabola which is the graph of the function. Since the axis of symmetry and the vertex of the parabola were found by the method of completing the square (page 52), it follows that this method can also be used to determine the maximum or minimum of the quadratic function as well as the corresponding value of the independent variable. This method, and an alternative method which makes use of the discriminant, are illustrated in the solutions of the following example.

EXAMPLE 1. Find the minimum value of $2x^2-5x+4$ and the corresponding value of x.

First solution, by completing the square:

$$\begin{aligned}2x^2-5x+4&=2\left(x^2-\frac{5}{2}x+\frac{25}{16}\right)+4-\frac{25}{8}\\&=2\left(x-\frac{5}{4}\right)^2+\frac{7}{8}.\end{aligned}$$

Now the square of 0 is 0 and the square of any other real number is positive. Hence the least value which $2\left(x-\frac{5}{4}\right)^2$ can have is 0 and it has the value 0 when $x=\frac{5}{4}$. It follows that the least value of $2x^2-5x+4$ is $\frac{7}{8}$, and the function takes this value when $x=\frac{5}{4}$.

Second solution, by using the discriminant:

Denote the function by y. Then,

$$2x^2-5x+4-y=0.$$

Since x is real, it follows (sec. 19) that the discriminant is positive or zero; that is,

$$25-8(4-y) \geqq 0$$

or

$$8y-7 \geqq 0.$$

Hence $y \geqq 7/8$, and therefore the least value of y is 7/8. The value of x which gives y its minimum value is now found by substituting 7/8 for y. Then

$$2x^2-5x+4=\frac{7}{8},$$

from which

$$16x^2-40x+25=0,$$

or

$$(4x-5)^2=0,$$

and

$$x=\frac{5}{4}.$$

It appears that when the maximum or minimum value of a quadratic function and the corresponding value of x are wanted, the first method of solution is more advantageous than the second. On the other hand, the second method has the advantage that it applies as well to certain other classes of functions (see sec. 23).

Example 2. A caterer agrees to provide a banquet for a club of 160 members on the following terms. For an attendance of 100 or fewer guests, the cost is to be \$2 each but, if the attendance exceeds 100, the cost for each guest is to be reduced by as many cents as there are guests over 100. What number of guests will give the caterer the greatest gross receipts?

If the attendance at the banquet is $100+x$, then each guest pays $200-x$ cents. The gross receipts are then

$$(100+x)(200-x)=20{,}000+100x-x^2$$

cents. The value of x for which this is a maximum is found by completing the square:

$$20{,}000-(x^2-100x)=22{,}500-(x-50)^2.$$

The greatest value of $-(x-50)^2$ is zero and the corresponding value of the quadratic is 22,500. Hence the receipts are greatest when $x=50$, that is, for 150 guests. For this number the caterer receives \$225.00.

The steps involved in finding maxima or minima in many problems can now be summarized.

To find a maximum or a minimum,

(1) *select a suitable independent variable,*

(2) *express as a function of this variable the quantity to be a maximum or a minimum,*

(3) *use one of the methods of Example 1.*

EXERCISES

1. For what values of x are the following functions negative? Sketch the graphs.

(*a*) $(x+4)(3x-5)$, (*b*) $10x-3x^2$, (*c*) $x^2-10x+25$.

2. For what values of x are the following functions positive? Sketch the graphs.

(*a*) $(3x+2)(x-3)$, (*b*) $-(x-2)^2$, (*c*) $25-4x^2$.

3. Investigate the sign of each of the following functions for real values of its variable:

(*a*) $3t^2-5t+3$, (*b*) $3x^2-2\sqrt{6}x+2$, (*c*) $20+7p-6p^2$.

4. Give the integral values of z for which the following functions are positive:

(*a*) $(z-7)(3-z)$, (*b*) $(3z-2)(5z+4)$, (*c*) $(z-1)(2-7z)$.

5. For each of the following functions, give the range of values of the independent variable for which the function is negative:

(*a*) $y^2+4y-21$, (*b*) $7x^2-x+1$, (*c*) $2-t-2t^2$.

6. Find the quadratic function of x which is positive except for values of x between 2 and $2\frac{1}{2}$, and which has the value 4 when $x=1$.

7. By inspection of each of the following functions, state the value of the variable which makes it a maximum or a minimum and give the maximum or minimum value.

(*a*) $7r^2-9$, (*b*) $9-7r^2$, (*c*) $(x-7)^2+4$, (*d*) $4-(x-7)^2$,

(*e*) $5(z+3)^2-32$, (*f*) $32-5(z+3)^2$.

8. Find the minimum value of each of the following and the corresponding value of x:

(*a*) $5x^2+4$, (*b*) x^2+x, (c) $x^2-2bx+2b^2$.

9. Find the maximum value of each of the following and the corresponding value of t:

$$(a)\ 8-t^2,\quad (b)\ 3+4t-t^2,\quad (c)\ 5t-3t^2.$$

10. Find the area of the largest rectangular field that can be enclosed with 200 rods of fence.

11. From a long strip of galvanized iron 12 inches wide, a tinsmith has to make a rectangular trough by creasing the metal along two lines parallel to the edges and bending each edge through a right angle. How far from the edges should the creases be made in order to obtain a trough with the maximum carrying capacity?

12. Suppose that an annual flowering plant grows to a height of 5 feet and that its rate of growth varies jointly as its height and the difference between its height and 5 feet. For what height is the growth most rapid?

13. A telephone exchange yields a net profit of $12 a year for each telephone if there are not more than 1000 subscribers. If the number of subscribers exceeds 1000, the profits on each telephone are decreased by as many cents as there are subscribers over 1000. What number of subscribers will give the maximum net profit for this exchange?

***23. Range of values of a quadratic fraction.** The problem of sec. 22 is an example of a type of problem of great interest and importance, namely, that of finding, for a given function, its maximum and minimum values when such exist. The methods most convenient for the solution of such problems are not algebraic. However, the methods of quadratics are useful in obtaining information regarding the real values which are possible for a fraction of the form $\dfrac{ax^2+bx+c}{a_1x^2+b_1x+c_1}$, in which a and a_1 are not both zero. The following examples will illustrate the method.

EXAMPLE 1. Find the range of values of the function

$$\frac{3x^2+4x+3}{x^2+x+1} \text{ for real values of } x.$$

Let $$\frac{3x^2+4x+3}{x^2+x+1}=y.$$

Then $$(3-y)x^2+(4-y)x+3-y=0.$$

If $y=3$, the coefficient of x^2 is zero and this equation is not quadratic. In this case, however, $x=0$ and hence 3 is a possible value of y. If $y \neq 3$, the equation is quadratic and, since x is real, the discriminant is positive or zero. Hence

$$(4-y)^2-4(3-y)^2 \geqq 0,$$

which gives $$(10-3y)(y-2) \geqq 0,$$

or $$-3y^2+16y-20 \geqq 0.$$

The quadratic function on the left of this inequality is positive only when y lies between 2 and 10/3 (sec. 21). Hence the range of values of y is indicated by $2 \leqq y \leqq 3\frac{1}{3}$. The maximum value of y is $3\frac{1}{3}$ and its minimum value is 2.

EXAMPLE 2. Find the range of values of $\dfrac{2x^2-6x-33}{x^2-3x-4}$ for real values of x.

Let $$\frac{2x^2-6x-33}{x^2-3x-4}=y.$$

Then $$(2-y)x^2-3(2-y)x-(33-4y)=0.$$

If $y=2$, this equation is not quadratic. Moreover, $y=2$ is seen by substitution to be an impossible value for y. If $y \neq 2$, the condition that x be real gives

$$9(2-y)^2+4(2-y)(33-4y) \geqq 0,$$

or $$(2-y)(150-25y) \geqq 0,$$

or $$y^2-8y+12 \geqq 0.$$

The quadratic on the left of this inequality is positive or zero for all values of y except those between 2 and 6 (sec. 21). As we have already seen that 2 is an impossible value, it follows that the range of possible values is $y<2$ and $y\geqq 6$.

EXERCISES ON CHAPTER III

A

1. Solve

(*a*) $x^2-4x+8=0$, (*b*) $6x^2-73x+105-0$,

(*c*) $25r^2+90r+81=0$.

2. Write the quadratic equation whose roots are

(*a*) -3 and 2, (*b*) $1/4$ and $-1/5$, (*c*) $-r$ and r,

(*d*) $m+1$ and $m-1$.

3. Without solving, determine the character of the roots of

(*a*) $8x^2+11-0$, (*b*) $3x^2-4\sqrt{3}x+4=0$,

(*c*) $(1+m)x^2+2(2+m)x+3+m=0$,

(*d*) $a^2x^2+ax+2=0$, where a is a real number.

4. Complete the square in each of the following functions:

(*a*) $4x^2+4x+9$, (*b*) $3x^2-4x$, (*c*) x^2+mx+n.

5. Find the quadratic function of x which takes the values 7, 5, and 15 when $x=0$, 1, and -1, respectively.

6. For each of the following equations, find the values of m for which the equation has real roots:

(*a*) $x^2-5x+m=0$, (*b*) $9x^2+mx+4=0$,

(*c*) $x^2-2(m-2)x+m^2=0$.

7. Draw the graph of the function $y=x^2-6x-7$. Use this graph (*a*) to describe the changes in y as x increases from -3 to 9, (*b*) to find the values of x for which y is negative, (*c*) to find the axis of symmetry, (*d*) to find the minimum value of the function.

8. From the graph of ex. 7, determine the real roots, if any, of the following equations:

$$(a)\ x^2-6x-7=0,\quad (b)\ x^2-6x-7=9,$$

$$(c)\ x^2-6x+5=0,\quad (d)\ x^2-6x+13=0.$$

9. Find the quadratic function $f(x)$ for which $f(0)=8$, $f(1)=5$, and $f(2)=6$.

10. Find, in the form $a+bi$, (*a*) the square, (*b*) the cube, (*c*) the reciprocal, of $4+3i$.

11. Find the minimum or maximum value and the corresponding value of x for

$$(a)\ x^2-6x+11,\quad (b)\ 7-5x-2x^2,\quad (c)\ 2x^2-5x+7.$$

12. Sketch the graph of each of the following functions and make a complete statement regarding its sign for all real values of x:

$$(a)\ (1+x)(3-x),\quad (b)\ 3x^2+17x-28,$$

$$(c)\ 4x^2+12x+10,\quad (d)\ -25+30x-9x^2.$$

13. If m and n are the roots of $x^2-5x+3=0$, find the value of (*a*) $\frac{m^2}{n}+\frac{n^2}{m}$, (*b*) m^4+n^4.

14. Divide 100 into two parts so that the product of the parts is a maximum.

15. Find the number which exceeds its square by the greatest possible amount.

B

16. Find the quadratic function which takes the values -9, 15, 79, when $x=2, 4, 6$, respectively.

17. Express as the product of linear factors:

(*a*) x^2+1, (*b*) x^2+x+1, (*c*) $4x^2+20x+19$.

18. Show that $2qx^2+(p+4q)x+p+2q=0$ has real roots if p and q are real.

19. Form the equation whose roots are obtained by adding 2 to each of the roots of $x^2+6x+11=0$.

20. For each of the following equations find the values of p for which the equation has real roots:

(*a*) $x^2+p=0$, (*b*) $x^2+px+p=0$, (*c*) $px^2+2x+p=0$.

21. Find the quadratic function of x whose graph crosses the x-axis where $x=1$ and where $x=5$ and whose ordinate where $x=3$ is 12.

22. Under what condition is x^2+px+q the square of a linear function of x?

23. The statement that the minimum value of x^2+1 is 1 seems to contradict the statement that the equation $x^2+1=0$ has two roots. Explain the apparent contradiction.

24. If m and n are the roots of $3x^2+7x-4=0$, form the equation whose roots are

(*a*) m^2 and n^2, (*b*) m^3 and n^3, (*c*) m^2n and mn^2.

25. State the property of the coefficients of $y=ax^2+bx+c$ which will ensure that its graph will (*a*) open in the positive direction of the y-axis, (*b*) pass through the origin, (*c*) touch the x-axis, (*d*) be relatively sharp and narrow near the vertex.

26. Find the condition on a, b, and c in order that one of the roots of $ax^2+bx+c=0$ may be (a) the reciprocal of the other, (b) double the other.

27. Three measurements of a certain quantity are made, giving the values a, b, and c. It is decided to use for the measure of the quantity the value of x which makes

$$(x-a)^2+(x-b)^2+(x-c)^2$$

a minimum. Show that this value is the arithmetic average of a, b, and c.

28. Use the identity $(x+y)^2-(x-y)^2=4xy$ to prove the following theorems: (a) If the product of two variables is constant, their sum is a minimum when they are equal. (b) If the sum of two variables is constant, their product is a maximum when they are equal.

29. If the cost of running a boat is $\frac{2}{5}x+\frac{90}{x}$ dollars per mile when the speed is x miles per hour, find the least cost per mile at which it can be run. (Hint: Use one of the theorems of ex. 28.)

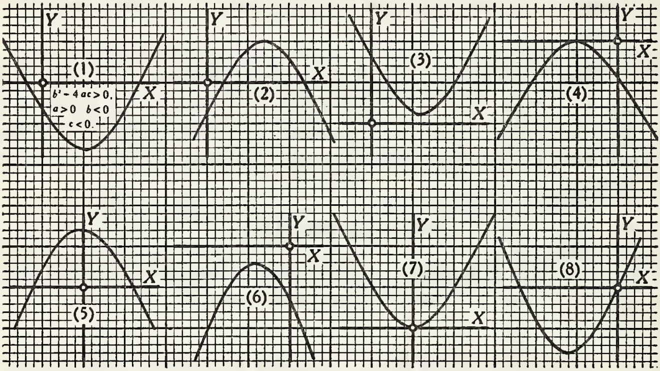
30. A rectangular field of 15 acres is to be enclosed by a fence and divided into two fields by a fence parallel to one side. Find the dimensions of the field which will require the least amount of fence.

31. A rectangular field is to be laid out along a straight river where no fence is needed along the river side. Find the dimensions of the largest field which can be fenced on the three sides with 120 rods of fence.

32. If a stone is projected vertically upwards with a speed of 100 feet per second, its distance above the ground in t seconds is given approximately by the formula $100t-16t^2$. Find the maximum height reached and the time it takes the stone to attain this height.

33. By completing the square, show that the function ax^2+bx+c has a maximum if $a<0$, and a minimum if $a>0$.

34. If each of the following figures is considered to be the graph of a function ax^2+bx+c, certain conditions on the coefficients can be deduced for each case. For the first figure, these conditions are given. Give the reasons which justify them and write the corresponding conditions for the other figures.



C

35. Find conditions in order that equations $ax^2+bx+c=0$ and $px^2+qx+r=0$ may have the same two roots.

36. Find a condition which must be satisfied by the coefficients in order that the equations

$$ax^2+bx+c=0 \text{ and } px^2+qx+r=0$$

may have one root in common.

37. Use the fact that the square of a real number cannot be negative to prove the following inequalities:

(*a*) $x^2+y^2 \geqq 2xy$, (*b*) $ax+by \leqq \sqrt{(a^2+b^2)(x^2+y^2)}$.

38. Find the range of values of $\dfrac{x^2+34x-71}{x^2+2x-7}$ for real values of x.

39. If $ax^2+bx+c=0$ is satisfied for more than two distinct values, of x, show that $a=b=c=0$.

40. If the roots of $ax^2+bx+c=0$ are in the ratio $m{:}n$, prove that $mnb^2=(m+n)^2ac$.

41. If ax^2+bx+c has the value k for three distinct values of x, show that $a=b=0$, and $c=k$.

42. State conditions which are necessary and sufficient in order that the roots of $ax^2+bx+c=0$ may be:

(i) real, different, and negative;

(ii) real and different and the numerically greater root positive.

(iii) real, equal, and positive;

(iv) equal in absolute value and opposite in sign;

(v) one root zero, the other negative;

(vi) both roots zero.

43. Find the range of values of $\dfrac{(2x+1)(x-3)}{(x-2)^2}$ for real values of x.

44. Criticize the statement: If b^2-4ac is a perfect square, $ax^2+bx+c=0$ has rational roots. From this point of view, examine the equation $x^2-2\sqrt{5}x+4=0$. State conditions which are sufficient in order that $ax^2+bx+c=0$ should have rational roots.

45. Show that from either of the equations

$$\sqrt{2x+8}-2\sqrt{x+5}=2$$

$$\sqrt{2x+8}-2\sqrt{x+5}=-2,$$

by steps involving two operations of squaring, the same quadratic equation can be deduced and that its roots satisfy the second equation but not the first.

46. Find the range of values of $\dfrac{x^2-2x}{x^2-4x+3}$ for real values of x.

47. Find the greatest value of $\dfrac{x^2-x+1}{x^2+x+1}$ for real values of x.

48. (*a*) Show that, when the radical signs are eliminated from the following equations by steps involving two operations of squaring, the same quadratic equation is obtained, of which one root satisfies the first equation, one the second and neither satisfies the third or fourth:

$$\begin{aligned}
&\text{(i)} & \sqrt{3x+7}+\sqrt{2x+3} &= \sqrt{12x+13},\\
&\text{(ii)} & \sqrt{3x+7}-\sqrt{2x+3} &= \sqrt{12x+13},\\
&\text{(iii)} & \sqrt{3x+7}+\sqrt{2x+3} &= -\sqrt{12x+13},\\
&\text{(iv)} & -\sqrt{3x+7}+\sqrt{2x+3} &= \sqrt{12x+13}.
\end{aligned}$$

(*b*) Insert the proper one of the words *necessary* and *sufficient* in each of the blanks in the following statements, where $Q=0$ denotes the quadratic equation mentioned in (*a*):
In order that (i) should hold, it is . . . . . but not . . . . . that $Q=0$; in order that $Q=0$, it is . . . . . but not . . . . . that (i) should hold.

49. Show that if a is real, the equation

$$(x+3)(1-ax)-(a+1)(x+3)=2$$

has two real roots, unless a is positive and less than 2.

50. Given that the roots of $ax^2+2bx+c=0$ are real and unequal, show that the roots of the following equation are imaginary:

$$(a^2-ac+2b^2)x^2+2b(a+c)x+(c^2-ac+2b^2)=0.$$

51. Find the restriction on the real values of m in order that, for all real values of x, $x^2-120+14m(x-17)+m^2(x-17)^2$ be not negative.

52. A quadratic function of x is positive except for the range of values $-2 \leqq x \leqq 1$, a second quadratic function of x is negative except for the range of values $-1 \leqq x \leqq 4$. Each function has the value 60 when $x=3$. For what other value of x are the functions equal?

53. (*a*) Show that every quadratic function of x can be written in the form

$$a(x-2)(x-3)+b(x-3)(x-1)+c(x-1)(x-2)$$

where a, b, and c are constants.

(*b*) Find a quadratic function, $Q(x)$, such that $\{Q(x)\}^5-x$ is divisible by $(x-1)(x-2)(x-3)$.

54. Show that, if a, b, p, and q are real numbers, so also are the roots of the equation

$$\frac{a^2}{x-p}+\frac{b^2}{x-q}=1.$$

55. A window has the form of a rectangle surmounted by an isosceles triangle whose height is 2/3 of its base. If the perimeter of the window is 60 feet, find the dimensions of the rectangle which will allow the window to admit a maximum amount of light.

CHAPTER IV

POLYNOMIALS AND ALGEBRAIC EQUATIONS

24. Polynomials. The linear and quadratic functions, already studied, have the forms

$$ax+b \text{ and } ax^2+bx+c, \text{ where } a\neq 0.$$

These expressions contain, besides the constants a, b, c, only positive integral powers of x, that is powers whose indices are positive integers. The linear function contains no power higher than the first and the quadratic function no power higher than the second. Consider now how to construct more general functions of the same type. An obvious method is to build up expressions containing only positive integral powers which, however, are not restricted to the first and second degrees. If a term containing x^3 is included, we have the general ***cubic function***

$$ax^3+bx^2+cx+d, \text{ where } a\neq 0.$$

This function is said to be of the ***third degree,*** since the index of its highest power is 3. The general ***quartic function*** is defined in a similar manner by the expression

$$ax^4+bx^3+cx^2+dx+e, \text{ where } a\neq 0.$$

This function is of degree 4.

It is clear that this definition can be extended to functions of the fifth, sixth, and higher degrees. This leads to the important concept of a *polynomial.*

DEFINITION. *A polynomial in a single variable x, or an **integral rational function** of x, is an expression of the form*

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0,$$

in which the coefficients are constants and n is a positive integer or zero. Although, according to the definition, the coefficients in a polynomial may be real or complex constants, we shall, in what follows, consider only polynomials whose coefficients are real numbers.

*The index of the highest power of the variable in a polynomial is called the **degree** of the polynomial.* In the order of their degrees, the five simplest classes of polynomials are constants*, followed by linear, quadratic, cubic, and quartic functions.

25. Cubic functions. As a first example, we consider the simplest cubic function, x^3. Setting $y=x^3$, we construct the following table of values:

x	-4	-3	-2	-1	0	1	2	3	4
y	-64	-27	-8	-1	0	1	8	27	64

---

*The polynomial given in the definition is of degree n provided $a_n \neq 0$. A constant different from zero is therefore a polynomial of degree 0. The constant 0 is a polynomial to which the definition of degree does not apply. In other words, the polynomial 0 has no degree.

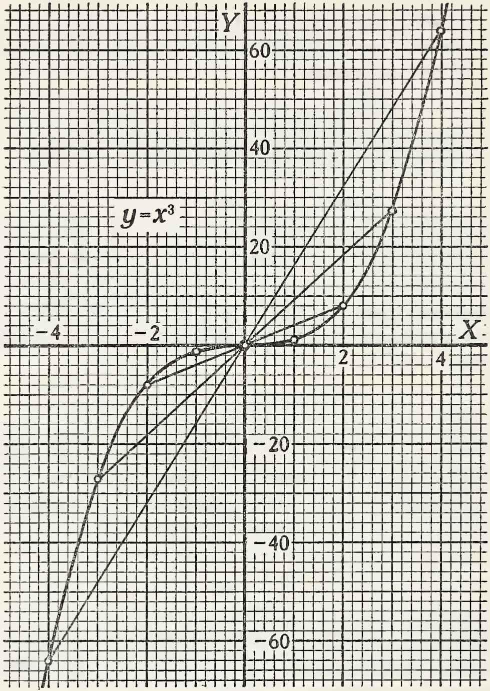


FIG. 16

From this table, the graph is constructed (fig. 16). As a point on the curve moves away from the origin, the curve becomes steeper and steeper. The curve is symmetrical with respect to the origin; that is, any straight line through the origin in the first and third quadrants cuts the curve in two points which are equidistant from the origin.

As an example of a cubic function with a different graph, we take

$$y = (x+1)(x-1)(x-3) = x^3 - 3x^2 - x + 3.$$

A table of values for this function is the following:

x	-4	-3	-2	-1	0	1	2	3	4	5	6
y	-105	-48	-15	0	3	0	-3	0	15	48	105

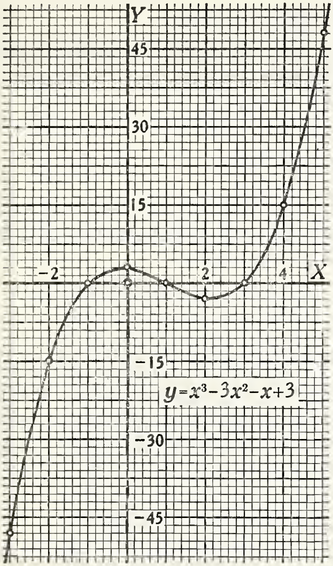


Fig. 17

The curve is shown in fig 17. When $x>3$ or <-3, the term x^3 is in magnitude the greatest term and since the numerical value of x^3 increases rapidly as x becomes large, it follows that the curve rises steeply on the right and falls steeply on the left. Any straight line parallel to the axis of x will cut the curve; hence any real number whatever will be a value of the function for a suitable value of x.

The two foregoing examples suggest some properties of the graph of the general cubic

$$y=ax^3+bx^2+cx+d.$$

In this function, the term ax^3 is the dominant term for numerically large values of x. This may be shown by writing the function in the form

$$y=x^3(a+b/x+c/x^2+d/x^3).$$

For large values of x, the terms b/x, c/x^2, and d/x^3 are small, so that a becomes much the largest term in the parenthesis and hence ax^3 the largest term of the polynomial. It follows that, if $a>0$ the graph rises indefinitely on the right and falls on the left. If $a<0$, the graph rises indefinitely on the left and falls on the right. In either case, the graph is cut at least once by any line parallel to the axis of x. Hence a cubic function takes all real values.

EXERCISES

1. Which of the following functions are polynomials and which are not? Justify your answers.

(*a*) x^5+1, (*b*) $x^{-5}+1$, (*c*) t^2-t,

(*d*) $(t^2-t)^{-1}$, (*e*) s^{100}, (*f*) $\dfrac{r^4-1}{r-1}$,

(*g*) x^2+x^{-2}, (*h*) x^2+2^x.

2. Give the degree and the coefficients of the powers of x in descending order for each of the polynomials:

(*a*) $(x^2-a^2)^2$, (*b*) $(x+a)(x+b)(x+c)$,

(*c*) $(x+r)^3-(x-r)^3$, (*d*) $(1+x^3)(1-x^2)$.

3. From the graph of fig. 17, read the values of x for which the function represented vanishes and the value of x for which it is -15. From the corresponding table, read the values of x which make the function take the values 48 and -105 respectively.

4. Draw the graphs of (*a*) $-x^3$, (*b*) $1+x^3$, (*c*) $(x-1)^3$.

5. From their graphs, describe the behaviour of the functions of 4(*a*) and 4(*c*).

6. Sketch the graph of a cubic function $f(x)$ which has the following properties: $f(0)=f(2)=f(4)=0$; $f(x)$ is positive for $0<x<2$ and for $x>4$; $f(x)$ is negative for $x<0$ and for $2<x<4$.

7. Without using tables of values, sketch the graphs of the following functions, showing the general character of the graphs and their intersections with the x and y axes.

(*a*) $(x-1)(x-4)(x-7)$, (*b*) $x(x^2-1)$,

(*c*) $x(1-x^2)$, (*d*) $x^2(x-1)$.

26. Cubic equations. If $f(x)$ is a cubic function, then $f(x)=0$ is a cubic equation. Thus, the cubic equation corresponding to the second example of sec. 25 is

$$x^3-3x^2-x+3=0,$$

or

$$(x+1)(x-1)(x-3)=0.$$

This equation has the roots -1, 1, and 3 which are the abscissas of the points in which the graph cuts the axis of x.

For any cubic equation,

$$ax^3+bx^2+cx+d=0,$$

the real roots are the abscissas of the points in which the graph of the corresponding function is cut by the axis of x. Since the

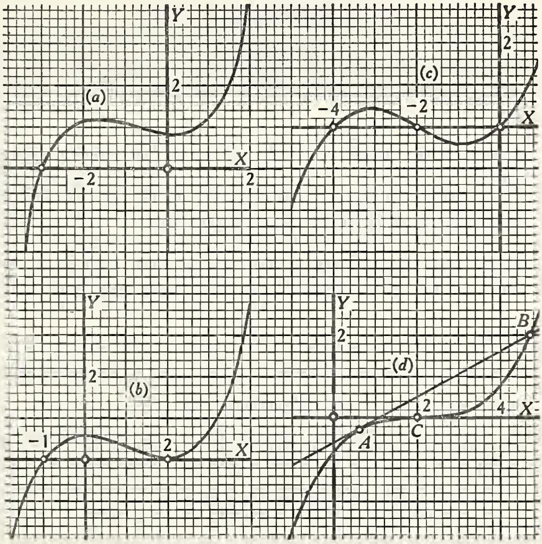


FIG. 18

x-axis always cuts the graph in at least one point, it follows that every cubic equation has at least one real root. From the graphs (fig. 18), we see that there are four possibilities with regard to intersections with the x-axis: (1) the x-axis may cut the curve in one and only one point (fig. 18a), (2) the x-axis may touch the curve in one point and cut it in another (fig. 18b), (3) the x-axis may cut the curve in three distinct points (fig. 18c), (4) the x-axis may touch the curve and cut it at the same point* (fig. 18d). In case (1), the corresponding cubic equation has just one real root. In case (2), the equation has three real roots of which two are equal; it is then said to have one single and one double root. In case (3), the equation has three distinct real roots. In case (4), the equation has three equal roots; it is then said to have a triple root.

EXERCISES

1. Give the roots of the following equations and describe how the graphs representing their left members are related to the x-axis:

$$(x+1)(x^2+1)=0;\quad x(x^2-1)=0;\quad x^3=0;\quad x^3+6x^2+12x+8=0.$$

2. If the cubic functions represented in fig. 18 are equated to zero, read the real roots of each equation.

3. Write the equations whose roots are: (*a*) 0, 1, and -1, (*b*) 1, 2, and 3, (*c*) 2, 2, and 2, (*d*) 1/2, 2/3, and 3/4.

4. Solve

$$(a)\ (2x+1)(3x+1)(4x+1)=0,\quad (b)\ (10x+3)(x^2+9)=0,$$
$$(c)\ x^3-16x=0.$$

---

*The fact that a point on a curve may be at the same time a point of contact and also a point of intersection with the x-axis may seem more real as a result of the following argument. Let a tangent (fig. 18d) touch the curve at A and cut it at B. As the point of contact A moves to the position C, B also moves to C. The tangent then becomes the axis of x, which both touches and cuts the curve at C.

5. What property has the graph of any polynomial of odd degree from which you may conclude that the corresponding equation has at least one real root?

6. What property has the graph of a polynomial of even degree from which you many conclude that the corresponding equation has either no real roots or an even number?

27. The Remainder Theorem. Consider the division of a polynomial $f(x)$ by another polynomial $P(x)$ of the same or lower degree. The process of division terminates when a remainder is obtained which is either zero or of lower degree than the divisor. Denote the quotient by $Q(x)$ and the remainder by $R(x)$. The work may be set down in brief form as follows:

$$\begin{array}{rcl} P(x)\ \big) & f(x) & \big(\ Q(x) \\ & \underline{P(x)\cdot Q(x)} & \\ & R(x) & \end{array}$$

This shows that

$$f(x)=P(x)\cdot Q(x)+R(x). \tag{1}$$

This relation is true for all values of x. It will be referred to as the *division relation,* and may be recalled by the statement:

dividend=divisor×quotient+remainder.*

When the divisor $P(x)$ is the linear function $x-a$ this formula yields the following useful result.

THE REMAINDER THEOREM. *If a polynomial $f(x)$ is divided by $x-a$, the remainder is $f(a)$.*

If the divisor is $x-a$, the division is carried out until a constant remainder R is obtained. Then the division relation becomes

$$f(x)=(x-a)Q(x)+R. \tag{2}$$

---

*This relation was known and used by Euclid, the celebrated author of the *Elements* of geometry, about 300 B.C.

This equation being true for all values of x, we may set $x=a$ and obtain

$$f(a)=0 \cdot Q(a)+R.$$

Therefore $$f(a)=R,$$

which proves the theorem.

COROLLARY. THE FACTOR THEOREM. *If* $f(a)=0$, *then* $x-a$ *is a factor of* $f(x)$.

For, in that case, $R=0$ and (2) becomes $f(x)=(x-a)Q(x)$. The converse of the factor theorem is also true, for, if $x-a$ is a factor of $f(x)$, it follows that $f(x)$ vanishes when $x=a$.

EXAMPLE 1. Find the remainder when x^3-5x^2+7x-4 is divided by $x-a$.

> According to the remainder theorem, the required remainder is a^3-5a^2+7a-4. The student will find it instructive to carry out the division and check his remainder with this formula.

EXAMPLE 2. Determine the value of a for which x^3-5x+a is exactly divisible by $x+3$.

> $x+3$ is a factor of the given polynomial if, and only if, $(-3)^3-5(-3)+a=0$; that is, if
>
> $$-27+15+a=0, \text{ or if } a=12.$$

EXAMPLE 3. Find a quadratic function of x which vanishes when $x=-2$, and which takes the values -6 and 4 when $x=1$ and 2, respectively.

> Since the function vanishes when $x=-2$, it contains the factor $x+2$. The other factor is a linear function, say $ax+b$.

Then

$$f(x)=(x+2)(ax+b).$$

Hence $f(1) = 3(a+b) = -6$

and $f(2) = 4(2a+b) = 4.$

It follows that

$$a+b = -2,$$
$$2a+b = 1,$$

from which $a = 3$ and $b = -5$. The required function is therefore

$$(x+2)(3x-5) = 3x^2+x-10.$$

EXERCISES

1. By actual division, find the remainder when

$$2x^3-5x^2+4x-7 \text{ is divided by } x+a$$

and check the result by the remainder theorem.

2. Find the remainder when $2x^3-3x^2+4x-7$ is divided by (*a*) x, (*b*) $x-2$, (*c*) $x+3$.

3. Find the remainder when $x+5$ is divided into (*a*) x^3, (*b*) x^4-625, (*c*) $2x^3+4x^2-9x+5$.

4. Show that the following statements are equivalent: (*a*) m is a root of $f(x)=0$; (*b*) the graph of $f(x)$ crosses the x-axis at the point $x=m$; (*c*) $x-m$ is a factor of $f(x)$; (*d*) when $f(x)$ is divided by $x-m$ the remainder is 0.

5. Use the division relation to show that, when a polynomial $f(x)$ is divided by $mx-a$, the remainder is $f(a/m)$.

6. Find the remainder when (*a*) x^4-5x^2+8 is divided by $x+3$; (*b*) $3x^3-7x^2+4x-9$ is divided by $x-4$; (*c*) x^3-6x+7 is divided by $3x+2$.

7. Determine whether $x-2$ or $x+3$ are factors of

$$(a)\ x^3-2x+21, \quad (b)\ x^4+x^3-2x^2+4x-24.$$

8. Show that a polynomial is divisible by $x-1$ when the sum of its coefficients is zero.

9. (*a*) State a condition on the coefficients of ax^3+bx^2+cx+d in order that it may be divisible by $x+1$. (*b*) Generalize this condition for any polynomial.

10. Use the factor theorem to show that (*a*) $x-a$ is a factor of x^m-a^m for all positive integral values of m; (*b*) $x+a$ is a factor of x^m-a^m if m is an even positive integer; (*c*) $x+a$ is a factor of x^m+a^m if m is an odd positive integer.

11. (*a*) What value of k will make x^3-6x^2+kx+2 divisible by (i) $x-1$, (ii) $x+2$? (*b*) What value of m will make $3x-2$ a factor of $3x^3+mx^2-2x-2$?

12. What value of k will make $2x^3+x^2-3kx+7$ yield a remainder 1, when divided by $x+2$?

28. **Algebraic equations.** Linear, quadratic and cubic equations are particular types under the general class of *algebraic equations.*

DEFINITION. *If $f(x)$ is a polynomial of degree $n>0$, then $f(x)=0$ is an algebraic equation of degree n.*

It follows that the general algebraic equation of degree n is

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0,\ a_n\neq 0,$$

in which the coefficients are here assumed to be real numbers. We have seen that the root of a linear equation is always real, and that the two roots of a quadratic equation are either real or imaginary. As it was found necessary to extend the number system in order that all quadratic equations should have roots, it might be supposed that a further extension of the number system would be necessary in order that equations of higher degree should have roots. Fortunately, however, this is not the case. The system of complex numbers (including as a sub-system the real numbers) turns out to be sufficient for the solution of algebraic equations of whatever degree. The theorem on which this conclusion is based is the following.

*28.1. The fundamental theorem of algebra. Every algebraic equation has a root, real or imaginary.**

A proof of this theorem is not easy to give and will be omitted from this book.† The experience which a student will acquire in handling equations will probably convince him of the truth of the theorem. We proceed to use it in the proof of the following theorem.

28.2. THEOREM. *A polynomial of degree n can be expressed as the product of n linear factors.*

Let $P_n(x)$ be a polynomial of degree $n(n>0)$. Then $P_n(x)=0$ is an algebraic equation which, according to the fundamental theorem of algebra, has a root. Denote a root by m_1 so that $P_n(m_1)=0$. Since $P_n(x)$ vanishes when $x=m_1$, it follows from the factor theorem that $x-m_1$ is a factor of $P_n(x)$. The other factor is a polynomial of degree $n-1$, which may be denoted by $P_{n-1}(x)$. Then

$$P_n(x) = (x-m_1)P_{n-1}(x).$$

The reasoning which has just been used for $P_n(x)$ applies also to $P_{n-1}(x)$. Hence, if $n-1>0$,

$$P_{n-1}(x)=(x-m_2)P_{n-2}(x),$$

where m_2 is a root of the equation $P_{n-1}(x)=0$, and $P_{n-2}(x)$ is a polynomial of degree $n-2$. Substituting the value of $P_{n-1}(x)$ from the second equation into the first, we get

$$P_n(x)=(x-m_1)(x-m_2)P_{n-2}(x).$$

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*The theorems of 28.1, 28.2, and 28.3 hold whether the coefficients are real or complex numbers. Students are advised to gain an understanding of the content of the theorems of this section, without memorizing their proofs.

†This theorem was first proved by Carl Friedrich Gauss (1777-1855), called "the prince of mathematicians", who gave four distinct proofs of it.

By successively repeating this operation, we obtain finally

$$P_n(x)=(x-m_1)(x-m_2)\cdots(x-m_n)P_0(x),$$

where $P_0(x)$ is a polynomial of degree 0, which is therefore a constant k, not zero. That is,

$$P_n(x)=k(x-m_1)(x-m_2)\cdots(x-m_n).$$

The constant k can be multiplied into any one of the n factors and hence $P_n(x)$ is the product of n linear factors.

It is to be noticed that the numbers $m_1, m_2, \cdots, m_n$, need not all be distinct, and therefore the n factors need not all be distinct. For example; $(x-2)^4(x+3)^3$ is a polynomial of degree 7 of which four factors are identical and the remaining three identical. Also, $(x-a)^n$ is a polynomial of degree n, whose n factors are all identical.

Although the theorem just proved states that every polynomial has linear factors, it gives no indication of a way of factoring the polynomial unless the roots of the corresponding equation are known. It should be noticed that, for many purposes, the factored form is more convenient than the expanded form and that, if a polynomial is presented in factored form, it is often a waste of labour to multiply out the factors. One advantage of the factored form is that it exhibits the roots of the corresponding equation; this is pointed out in the following sub-section.

28.3. THEOREM. *An algebraic equation of degree n has exactly n roots.* This is a corollary from the preceding theorem. If the equation of degree n is $P_n(x)=0$, it can be written in the form

$$k(x-m_1)(x-m_2)\cdots(x-m_n)=0, \tag{3}$$

which shows that the equation has the n roots $m_1, m_2, \cdots, m_n$. It remains to prove that the equation cannot have more roots than these. Suppose, if possible, that it has a root l different from $m_1, m_2, \ldots, m_n$. Then (3) must be satisfied when l is substituted for x:

$$k(l-m_1)(l-m_2)\ldots(l-m_n) = 0.$$

Each factor in the left member of this equation is different from 0, and hence their product cannot be zero. Therefore there can be no root l distinct from the m's. It follows that the equation has exactly n roots.

Again, it is to be noticed that these roots need not all be distinct. If two but not more than two of the roots are equal to m_1, then m_1 is a double root or a root of multiplicity 2. In the same way we may have triple roots, quadruple roots, or, in general, roots of multiplicity r. To make up the count of n roots for an equation of degree n, these roots must be counted according to their multiplicities; that is, a root of multiplicity r counts as r roots.

COROLLARY 1. *If the equation*

$$a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0 = 0 \tag{4}$$

has more than n roots, then $a_n=a_{n-1}=\ldots=a_0=0$.

Suppose that (4) has more than n roots. If $a_n\neq 0$, then, by 28.3, (4) has only n roots; hence $a_n=0$. Then if $a_{n-1}\neq 0$, (4) has only $n-1$ roots; hence $a_{n-1}=0$, etc. By this reasoning, it follows that all the coefficients are zero.

COROLLARY 2. *If two polynomials of the nth degree in x are equal for* $n+1$ *or more values of x, the coefficients of like powers of x in the two polynomials are equal.*

For, by equating their difference to zero, we have an equation of the form (4) which is satisfied by more than n values of x. Hence all its coefficients are zero and therefore the coefficients of like powers of x in the two polynomials are equal. Thus the two polynomials are identical, and the equation obtained by equating them is true for all values of x. Such an equation is called an ***identity***.

Example 1. Find a cubic function of x which vanishes when $x=-1$ and when $x=2$, and which takes the values 2 and 12 when $x=1$ and 3, respectively.

Since the function vanishes when $x=-1$ and when $x=2$, it contains the factors $x+1$ and $x-2$. Since the function is of the third degree, it has one other linear factor, say $ax+b$ (sec. *28.2*).

Then $$f(x)=(x-2)(x+1)(ax+b).$$

Hence $$f(1)=-1\times 2\times(a+b)=-2a-2b=2,$$

and $$f(3)=1\times 4\times(3a+b)=12a+4b=12.$$

The solution of these equations for a and b gives $a=2$, $b=-3$. The required function is therefore

$$(x-2)(x+1)(2x-3)=2x^3-5x^2-x+6.$$

Example 2. If ax^3+3x^2+bx+7 has the same values as $6x^3-cx^2-5x+d$ when x has the values 0, 1, 2, 3, respectively, find the values of a, b, c, and d.

Since the functions are of degree 3 and are equal for 4 values of x, it follows (corollary 2) that the two functions are identical; that is, the corresponding coefficients are equal. Hence

$$a=6,\ b=-5,\ c=-3,\ d=7.$$

Example 3. Express $2x^3-3x^2-5x+2$ as a polynomial in powers of $x-1$.

Set $y=x-1$; then $x=y+1$. Hence

$$2x^3-3x^2-5x+2=2(y+1)^3-3(y+1)^2-5(y+1)+2,$$

which simplifies to $2y^3+3y^2-5y-4$.

It follows that the given polynomial, expressed in powers of y, or $x-1$, is

$$2(x-1)^3+3(x-1)^2-5(x-1)-4.$$

EXERCISES

1. Which of the following are *algebraic equations* according to the definition? Of those which are not algebraic, which have roots?

(*a*) $x^2+2=7x$, (*b*) $(x+1)^2-(x-1)^2=4x$,

(*c*) $2^x=8$, (*d*) $(x+3)^2=x(x+6)$.

2. By means of factoring, express each of the following polynomials as the product of linear factors:

(*a*) x^3-9x, (*b*) x^4-5x^2+4, (*c*) x^4-8x^2-9.

Is it always possible to express a polynomial as a product of *real* linear factors?

3. The cubic function $x^3+2x^2-3x-10$ vanishes when $x=2$. Make use of this fact to express the function as the product of three linear factors.

4. If $ax^2+bx+c=p(x+1)^2+q(x+1)+r$ for three values of x, find p, q, r, in terms of a, b, c.

5. Is $2^x=0$ an algebraic equation? By testing, find whether this equation is satisfied by any real value of x, positive, negative, or zero.

6. Find the cubic function of x which becomes 0 when $x=-1, \frac{1}{2}, 2$, and which has the value 4 when $x=0$.

7. For what value of m is each of the following equations an identity?

(*a*) $3x^2-mx+12-3(x+2)^2=0$,

(*b*) $(mx+2)^2+4x(x-2)-8x^2-4=0$.

8. In each case, form the equation whose roots with their multiplicities are as follows:

(*a*) 3, -5, $\frac{1}{2}$, all single roots;

(*b*) a double root 1, and single roots, 0, $-\frac{2}{3}$;

(*c*) a triple root 0, a double root -2,
and single roots 1, -1.

9. Find a cubic function of x which vanishes when $x=1$ and when $x=3$, and which takes the values -3 and 48 when $x=2$ and 5, respectively.

10. Express $4x^3-6x^2-3x+10$ as a polynomial in powers of $x-1$.

11. Express z^3+8 as a polynomial in powers of $z+2$.

29. Solution of certain types of algebraic equations. Algebraic equations of higher degree than the second present much greater difficulty in their solution than those of the first and second degrees. The general question of the solution of cubic and quartic equations must be left to more advanced courses. There are, however, some cases in which equations may be reduced so as to depend on quadratic equations, and others in which some peculiarity in the equation enables us to detect a root. For the most part, we shall confine attention to equations in which the coefficients are rational numbers. If, in such an equation, some of the coefficients are fractions, by means of a suitable multiplier the equation can be replaced by one in which the coefficients are integers.

29.1. Equations reducible to quadratics. If the unknown in a quadratic equation is $f(x)$, then the solution of the quadratic yields two roots, m and n, for $f(x)$. The solution of the equations $f(x)=m$ and $f(x)=n$ then gives the roots for x.

Example 1. Solve $(x^2-3x)^2+(x^2-3x)-20=0$.

If y is substituted for x^2-3x, we have

$$y^2+y-20=0,$$

whence $$(y+5)(y-4)=0$$

and $$y=4 \text{ or } -5.$$

Hence $x^2-3x=4$, from which $x=4$ or -1,

or $x^2-3x=-5$, from which $x=\dfrac{3\pm\sqrt{-11}}{2}$.

Thus, the four roots of the given equation are

$$4,\ -1,\ \frac{3\pm\sqrt{-11}}{2}.$$

Example 2. Solve $(x+1)(x+2)(x-6)(x-7)=180$.

By combining the first factor with the third and the second with the fourth, we get

$$(x^2-5x-6)(x^2-5x-14)=180.$$

This equation is reduced to a quadratic by various substitutions. For example,

set $$x^2-5x-6=y.$$

Then $$y(y-8)=180,$$

whence $$y^2-8y-180=0,$$

and $$y=18 \text{ or } -10.$$

If $y=18$, then $x^2-5x-6=18$ or $x^2-5x-24=0$,

whence $x=8$ or -3.

If $y=-10$, then $x^2-5x-6=-10$ or $x^2-5x+4=0$, whence $x=1$ or 4.

Hence the roots of the given quartic equation are 1, 4, -3, and 8.

EXAMPLE 3. Solve $30x^4-13x^3-230x^2-13x+30=0.$

The peculiarity of this equation is the symmetrical form of the polynomial as regards its coefficients. If m is a root, it is easily verified that $1/m$ is also a root. Equations of this form are called *reciprocal equations.* Dividing by x^2, we get

$$30x^2-13x-230-\frac{13}{x}+\frac{30}{x^2}=0,$$

or
$$30\left(x^2+\frac{1}{x^2}\right)-13\left(x+\frac{1}{x}\right)-230=0.$$

Set
$$x+\frac{1}{x}=y;\text{ then } x^2+\frac{1}{x^2}+2=y^2.$$

Hence
$$30(y^2-2)-13y-230=0,$$

or
$$30y^2-13y-290=0,$$

which gives, by factoring,

$$(3y-10)(10y+29)=0.$$

This gives two equations for x. The first is

$$3x+\frac{3}{x}-10=0,$$

or
$$3x^2-10x+3=0,$$

or
$$(x-3)(3x-1)=0,\text{ whence } x=3 \text{ or } \frac{1}{3}\cdot$$

The second equation is

$$10x+\frac{10}{x}+29=0,$$

or
$$10x^2+29x+10=0,$$

or
$$(5x+2)(2x+5)=0,\text{ whence } x=-\frac{2}{5}\text{ or }-\frac{5}{2}.$$

Hence, the four roots are $3, \frac{1}{3}, -\frac{2}{5}$, and $-\frac{5}{2}$.

29.2 In some cases, a root of an equation can be discovered by inspection or by trial. In the search for roots of an equation $f(x)=0$, the following facts are of assistance:

(*a*) *If the constant term of* $f(x)$ *is* 0, *then* 0 *is a root of the equation.*

(*b*) *If the sum of the coefficients of* $f(x)$ *is* 0, *then* 1 *is a root of the equation.*

(*c*) *If* m *is a root of* $f(x)=0$, *then* $x-m$ *is a factor of* $f(x)$, *and conversely.* When a root m is discovered, the factor $x-m$ can be divided out from $f(x)$ and the equation replaced by one of lower degree.

(*d*) *If* $f(x)$ *has integral coefficients, then the integral roots, if any, of* $f(x)=0$ *are factors of the constant term of* $f(x)$.

For, suppose the equation is

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0,$$

where the a's are integers. By transposing a_0, we get

$$x(a_nx^{n-1}+a_{n-1}x^{n-2}+\cdots+a_1)=-a_0.$$

If x is an integer, then the quotient obtained by dividing $-a_0$ by x is the number in parenthesis, which is an integer. Hence every integral root is a factor of a_0.

EXAMPLE 4. Solve $6x^3-7x^2-x+2=0$.

Since the sum of the coefficients is 0, 1 is a root. Dividing out $x-1$ from the left member, we get

$$6x^2-x-2=0,$$

or

$$(2x+1)(3x-2)=0.$$

The three roots of the cubic are therefore

$$1, -1/2, \text{ and } 2/3.$$

EXAMPLE 5. Solve $x^3-7x^2+8x+10=0.$

The only possible integral roots are exact divisors of 10, that is $\pm1, \pm2, \pm5$ and ±10. It is found by trial that the only one of these numbers which satisfies the equation is 5.

Dividing out $x-5$, we get

$$(x-5)(x^2-2x-2)=0,$$

whence

$$x=5 \text{ or } 1\pm\sqrt{3}.$$

EXERCISES

Solve:

1. $3(x^2-x)^2-24(x^2-x)+36=0.$
2. $(x^2+8x+15)(x^2+6x+8)-24=0.$
3. $3x^4-19x^2+28=0.$
4. $x^3+2x^2-7x+4=0.$
5. $3x^3-4x^2-27x-20=0.$
6. $x^3-x^2-4x-6=0.$
7. $12(x^4+1)-56(x^3+x)+89x^2=0.$
8. $2x^4-x^3-8x^2+4x=0.$
9. $(3x-5)(3x+1)^2(3x+7)+68=0.$
10. $6x^4-35x^3+56x^2-35x+6=0.$

30. Imaginary Roots. THEOREM. *For an algebraic equation with real coefficients, imaginary roots occur in pairs.*

We have seen that when the roots of a quadratic equation are imaginary they are of the form $p\pm iq$, where p and q are real numbers. We shall now show that, for an equation of any degree, the imaginary roots, if any, occur in pairs of this form. The number of imaginary roots of such an equation is therefore always an even number.

Suppose the equation $f(x)=0$ has an imaginary root $p+iq$. We wish to show that $p-iq$ is also a root. This will be true if $x-(p+iq)$ and $x-(p-iq)$ are factors of $f(x)$, or if $(x-p)^2+q^2$ is a factor of $f(x)$. Let $(x-p)^2+q^2$ be divided into $f(x)$. The remainder, if any, will be of the first degree in x or a constant, and since the coefficients of the dividend and of the divisor are real, those of the remainder will also be real. Denote the quotient by $Q(x)$ and the remainder by $rx+s$. The division relation yields the following identity:

$$f(x)=\{(x-p)^2+q^2\}\,Q(x)+rx+s.$$

In this identity, set $x=p+iq$. Then, since $f(p+iq)=0$, we have

$$0=0\cdot Q+r(p+iq)+s.$$

Therefore $rp+s+irq=0$.

Since p, q, r, and s are real numbers, it follows that $rp+s=0$, and $rq=0$. Since $p+iq$ is imaginary, $q\neq 0$. Hence, from the last equation, $r=0$ and therefore, from the preceding equation $s=0$. It follows that the remainder $rx+s$ is zero and the theorem is proved.

EXERCISES

(In the following exercises, all equations mentioned are to have real coefficients.)

1. State the possible numbers of real and of imaginary roots for a cubic, for a quartic, and for a quintic equation. Give an example of an equation to illustrate each case in your list.

2. Form the quadratic equation which has for one root $3-4i$.

3. Form the cubic equation which has for two roots $2/3$ and $2+i$.

4. Form the quartic equation which has for two roots $4i$ and $2-4i$.

5. Solve $x^4-2x^3-2x^2+8x-8=0$, given that one root is $1+i$.

6. Solve $3x^4-28x^3+96x^2-128x+33=0$, given that one root is $3-i\sqrt{2}$.

★31. Synthetic division. It follows from the remainder theorem that the result of substituting a number a in a polynomial $f(x)$ can be found by dividing $f(x)$ by $x-a$ and taking the remainder. This process has some advantage over that of direct substitution of a for x, an advantage arising from the fact that the work of division can be abbreviated so that only arithmetical operations appear. The method and the reasoning underlying it can be made clear by considering the division of the general cubic $a_3x^3+a_2x^2+a_1x+a_0$ by $x-m$. Denote the quotient, which is a quadratic, by $b_2x^2+b_1x+b_0$ and the remainder by R. Our problem is to determine the unknowns b_2, b_1, b_0, and R. By the division relation,

$$\begin{aligned} a_3x^3+a_2x^2+a_1x+a_0 &= (x-m)(b_2x^2+b_1x+b_0)+R \\ &= b_2x^3+(b_1-mb_2)x^2+(b_0-mb_1)x+R-mb_0. \end{aligned}$$

Since this is an identity, the coefficients of like powers of x are equal. Hence

$$\begin{aligned} b_2 &= a_3, \\ b_1-mb_2 &= a_2 \text{ and } b_1 = a_2+mb_2, \\ b_0-mb_1 &= a_1 \text{ and } b_0 = a_1+mb_1, \\ R-mb_0 &= a_0 \text{ and } R = a_0+mb_0. \end{aligned}$$

The work of finding b_2, b_1, b_0, and R can now be arranged as follows:

$$\begin{array}{lllll} a_3+ & a_2+ & a_1+ & a_0 & \underline{|\,m} \\ & mb_2+ & mb_1+ & mb_0 & \\ \hline b_2+ & b_1+ & b_0\;| & R & \end{array}$$

From the first column, b_2 is found (equal to a_3). The entry mb_2 is then made under a_2, and b_1 is found by adding the second column. Then the entry mb_1 is made under a_1, and b_0 is found by adding the third column. Finally, the entry mb_0 is made under a_0, and R is found by adding the last column.

In this process, known as synthetic division, the dividend is represented only by its coefficients. In a numerical case, the symbols b_2, b_1, b_0, and R are dispensed with, and the numerical values written for the sums of the columns. The process is illustrated in the following examples.

EXAMPLE 1. Divide x^3-4x^2+5x-7 by $x-3$.

The work is arranged as follows:

$$\begin{array}{rrrrl}
1 & -4 & 5 & -7 & \underline{|3} \\
 & 3 & -3 & 6 & \\
\hline
1 & -1 & 2\,| & -1 &
\end{array}$$

Quotient, x^2-x+2; remainder, -1.

EXAMPLE 2. If $f(x)=x^4+2x^3+4x-6$, find $f(-4)$.

This is accomplished by dividing the polynomial by $x+4$. The coefficient of x^2 in the dividend is 0. Hence, when we detach coefficients, the numerical work appears as follows:

$$\begin{array}{rrrrrl}
1 & 2 & 0 & 4 & -6 & \underline{|-4} \\
 & -4 & 8 & -32 & 112 & \\
\hline
1 & -2 & 8 & -28\,| & 106 &
\end{array}$$

Hence $f(-4)=106$.

EXERCISES

Use synthetic division in the following problems.

1. Divide (*a*) x^3-6x^2+5x-4 by $x+2$,

   (*b*) $2x^4+3x^3-x-2$ by $x-3$,

   (*c*) x^4+4 by $x-2$,

   (*d*) $3x^3-5x+2$ by $x+6$.

2. Given $f(x)=6x^3-2x^2+8x+4$, find the remainders when $f(x)$ is divided by

   (*a*) $x-4$, (*b*) $2x-6$, (*c*) $3x+2$.

3. Given $f(x)=2x^3+5x^2-6x+6$, find $f(-6)$ and $f(3/2)$.

4. Given $f(x)=x^4-11x^2+20$, find $f(-2.5)$ and $f(9)$.

5. Which of the following are factors of $5x^3-6x^2+11x-2$:

   (*a*) $x-2$, (*b*) $x-1/5$, (*c*) $x+2$?

6. Is 3 a root of the equation

$$x^5+6x^4-25x^3+14x^2-30x-90=0?$$

★32. **Location of real roots between pairs of consecutive integers.** Consider the problem of locating a non-integral real root of an equation $f(x)=0$. Imagine the graph of $f(x)$ to be drawn. We shall assume that it possesses a property which has been found to hold for linear, quadratic, and cubic curves, namely the property of being an unbroken curve; this may be expressed by saying that the curve is continuous at all its points. The abscissas of the points where the graph cuts the axis of x are the real roots of the equation. When the graph crosses the axis of x, the ordinate of a point on it changes sign; moreover, since the graph is an unbroken curve, the ordinate changes sign only when the curve crosses the axis of x. Stated in terms of equations, these facts give the following theorem: *If $f(x)$ is* *negative for $x=a$ and positive for $x=b$, then between a and b lies at least one root of $f(x)=0$.* In this statement, a may be either greater or less than b.

It does not follow that, if $f(a)$ and $f(b)$ have the same sign, there are no roots between a and b; in fact, there might be an even number of roots. As a first approximation to the real roots, it is convenient to locate them, if possible, between consecutive integers. If there is not more than one real root between any two consecutive integers, the roots can be located by finding the signs of $f(x)$ for successive integral values of x.

EXAMPLE. Show that the three roots of

$$f(x) = x^3-3x^2-12x+15 = 0$$

are real, and locate them between pairs of consecutive integers.

By synthetic division, we find the values in the following table:

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-3	19	23	15	1	-13	-21	-17	5

It follows that one root lies between -3 and -2, a second between 1 and 2, and the third between 4 and 5.

★**33. Approximate determination of a real root.** Suppose it is desired to approximate a root to a given number of figures. The computations involved are then likely to be laborious and various devices are used to facilitate the work. These devices are studied in books on the theory of equations. If, for a root which has been located between two consecutive integers, an approximation of two more figures is desired, the work may conveniently be carried out by a slight extension of the method of sec. 32 for locating the roots. This is illustrated in the following example.

EXAMPLE 1. Compute to three figures the root of the foregoing equation, which was found to lie between 4 and 5.

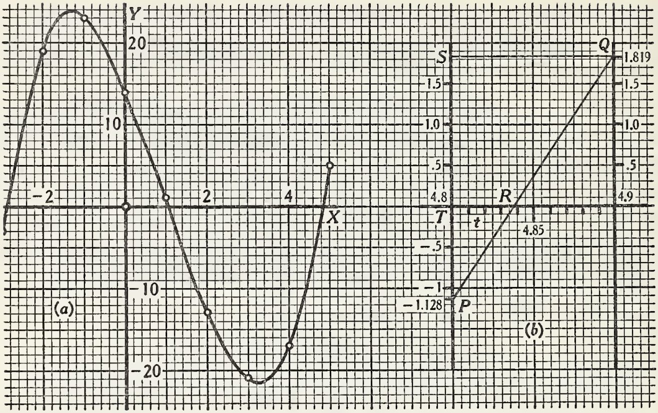

Fig. 19

The graph (fig. 19a) of $f(x)$ shows the root to be nearer to 5 than to 4; from it one might estimate the root to be about 4.8. By synthetic division, we find the values of $f(4.8)$ and $f(4.9)$ as follows:*

$$\begin{array}{rrrrl} 1 & -3 & -12 & +15 & \underline{|4.8} \\ & 4.8 & +\ 8.64 & -16.128 & \\ \hline 1 & +1.8 & -\ 3.36\ | & -\ 1.128 & \end{array}$$

$$\begin{array}{rrrrl} 1 & -3 & -12 & +15 & \underline{|4.9} \\ & +4.9 & +\ 9.31 & -13.181 & \\ \hline 1 & +1.9 & -\ 2.69\ | & +\ 1.819 & \end{array}$$

Hence $f(4.8) = -1.128$ and $f(4.9) = 1.819$. The root is therefore between 4.8 and 4.9 and is likely nearer to 4.8, since $f(4.8)$ is nearer to 0 than is $f(4.9)$.

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*More than two trials may be necessary to locate the first decimal place. If, for example, we try 4.7, we find $f(4.7) = -3.847$. This and the value of $f(4.8)$ indicate that the root is not between 4.7 and 4.8 but is greater than 4.8.

From these two results, the third figure in the root can be found by interpolation. Let P be the point on the graph at which $x = 4.8$, and Q the point at which $x = 4.9$. Between these two points the graph is approximately straight. The point R, at which the chord PQ crosses the x-axis provides a good approximation to the desired root. This chord is shown on a magnified scale in fig. 19b. From this figure the distance $t\,(=TR)$ is seen to be 4/10 of 0.1 or 0.04, correct to the first significant figure. This gives the approximation 4.84 for the root.

The method just described for the determination of t (and hence of the third figure of the root) requires a figure drawn carefully to scale. On the other hand, the value of t can be calculated from a rough sketch of this figure. For, from similar triangles,

$$\frac{t}{1.128} = \frac{QS}{PS} = \frac{0.1}{1.128+1.819} = \frac{0.1}{2.947},$$

whence $t = \dfrac{0.1128}{2.947} = 0.04$, to the first significant figure.

EXERCISES

1. Locate between pairs of consecutive integers the real roots of

(*a*) $x^3+2x^2-5x-5 = 0$, (*b*) $x^3+2x-5 = 0$,

(*c*) $2x^3+x^2+x+3 = 0$, (*d*) $x^3-3x^2+2x+1 = 0$.

2. Show that the equation $x^3+x^2+x-6 = 0$ has a root between 1 and 2 and find this root correct to three figures.

3. Locate and compute to three figures the positive root of $x^3+2x^2-6x-5 = 0$.

4. If x is a cube root of 3, then $x^3-3=0$. Hence find, to three figures, the real cube root of 3.

5. Find, to three figures, the cube root of 35.

33.1. *Problems leading to cubic equations.* If a problem gives rise to an equation of higher degree than the second, it may be necessary to approximate to a particular root. The method of this section may then be applied.

Example 2. A rectangular block of ice has the dimensions 1 ft. by 2 ft. by 2 ft. If, when the ice melts, each edge is diminished by the same amount, find to the nearest hundredth of a foot, the decrease in each edge when half the ice in the block is melted.

If x is the required decrease in each edge, the condition in the problem is expressed by the equation

$$(1-x)(2-x)(2-x) = 2,$$

or

$$x^3-5x^2+8x-2 = 0.$$

From the problem one might estimate a root of this equation to be in the neighbourhood of half a foot or less. The values of the left member for $x = 0.5$, 0.4, and 0.3 are found by synthetic division as shown below:

$$\begin{array}{rrrrl}
1 & -5 & +8 & -2 & \underline{|0.5} \\
 & 0.5 & -2.25 & 2.875 & \\
\cline{1-4}
1 & -4.5 & +5.75\,| & +0.875 &
\end{array}$$

$$\begin{array}{rrrrl}
1 & -5 & +8 & -2 & \underline{|0.4} \\
 & 0.4 & -1.84 & 2.464 & \\
\cline{1-4}
1 & -4.6 & +6.16\,| & +0.464 &
\end{array}$$

$$\begin{array}{rrrrl}
1 & -5 & +8 & -2 & \underline{|0.3} \\
 & 0.3 & -1.41 & +1.977 & \\
\cline{1-2} \cline{3-4}
1 & -4.7 & +6.59\,| & -0.023 &
\end{array}$$

Hence $f(0.5)=0.875$, $f(0.4)=0.464$, and $f(0.3)=-0.023$. The root is between 0.3 and 0.4 and very close to 0.3. In fact, using the notation and method of the previous example, we find

$$\frac{t}{0.023}=\frac{0.1}{0.487}$$

from which $t=0.00$ to two places of decimals. Hence the required root is 0.30.

EXERCISES

In each problem find the answer correct to the second decimal place.

1. The edges of a rectangular solid are 2, 3, and 4. By what fixed amount must each edge be increased in order to double the volume?

2. The volume of a segment of a sphere cut off by a plane is $\pi x^2\left(r-\frac{x}{3}\right)$, where r is the radius of the sphere and x is the depth of the segment. If a sphere of wood floats in water with 2/3 of its volume immersed, find the height of the highest point of the sphere above the surface of the water.

EXERCISES ON CHAPTER IV

A

1. Give the remainder when (*a*) x^5 is divided by $x-2$; (*b*) x^{20} is divided by $x+1$; (*c*) $(x-9)^{100}$ is divided by $x-10$.

2. Find the values of a and b which make ax^3-4x^2+bx-6 exactly divisible by $x-3$ and by $x+1$.

3. Form the equation whose roots are

(*a*) $-3, 2, 0$; (*b*) $-4, \frac{-2\pm i}{3}$; (*c*) $2\pm\sqrt{3}, 2\pm i\sqrt{3}$.

4. Given that 3/5 is a root of the equation

$$5x^3-13x^2+x+3=0,$$

find the other roots.

5. (*a*) Solve $(2x^2-3x-3)(2x^2-3x-5)=24$.

(*b*) Express $(2x^2-3x-3)(2x^2-3x-5)-24$ as the product of four linear factors.

6. Sketch the graphs of the following functions:

(*a*) $x(x-4)^2$; (*b*) $(x+1)^2(x-3)^2$; (*c*) $(x^2-1)(x^2-4)$; (*d*) $x^3(x-1)$.

7. Form the cubic equation which has -2 as a double root and 1 as a single root.

8. The cubic equation

$$x^3+3x^2-14x-12=0$$

has the root 3. Find its other two roots.

9. Form the cubic equation with real coefficients which has $3-2i$ and 1 as two of its roots.

10. Solve $x^4-11x^2-60=0$.

11. Solve $x^3-x^2+x=0$.

12. Factor the following polynomials into real linear and quadratic factors and sketch their graphs:

(*a*) x^3-8; (*b*) x^5-x^2; (*c*) x^4-16.

13. If the functions $2ax^2+5x-4c$ and $3x^2-2px+8$ are equal when x has the values 100, 200, and 300, find a, c, and p.

14. What values of x satisfy the equation

$$(x+3)(x-4)+(x+1)(2-x)+10=0.$$

Give your reasons.

15. Find the cubic function of x which vanishes when $x=-1, 1, 4$, and for which $f(2)=-12$.

B

16. Show that, if all the terms of $f(x)$ have the same sign, the equation $f(x)=0$ cannot have a positive root.

17. Form the equation which has the roots l, m, n. Hence, for the equation $x^3+ax^2+bx+c=0$, determine (a) the sum of the roots, (b) the product of the roots.

18. If k, l, m, n are the roots of $x^4+ax^3+bx^2+cx+d=0$, express a, b, c, d in terms of k, l, m, n.

19. Express t^3+t^2+t+1 as a polynomial in powers of $t+1$.

20. A quartic function of x vanishes when x equals $-2, -1, 1$, or 2 and has the value 8 when $x=0$. Find the function.

21. Find the cubic function of x whose values for

$x=1, -2, 3, 4$ are 0, 0, 40, 144, respectively.

22. By inspection of the coefficients, detect one root of the equation

$$x^3-6x^2+8x-3=0.$$

Hence solve the equation completely.

23. Solve $x^4+4x^3-10x^2+4x+1=0$.

24. Solve $3x^4+11x^3+18x^2+13x-3=0$, given that one root is $-1+\sqrt{-2}$.

25. Solve $(x+2)(x+3)(2x-1)(2x-3)=540$.

26. Solve $x^4+2x^3-11x^2+4x+4=0$.

27. Given that $x^4-6x^3+12x^2-10x+3=0$ has a triple root, solve the equation.

28. The roots of the cubic equation

$$x^3-8x^2+5x-3=0$$

are all roots of the quartic equation

$$x^4-6x^3-11x^2+7x-6=0.$$

Verify this statement and find the fourth root of the quartic equation.

29. Show that, if an algebraic equation contains only even powers of x, its roots can be taken in pairs such that those of each pair differ only in sign.

30. State a theorem corresponding to that of ex. 29 for an equation which contains only odd powers of x.

31. Find two real linear factors of the polynomial

$$f(x)=2x^4+7x^3+2x^2-5x-6,$$

and hence solve the equation $f(x)=0$.

32. Show that, if x^2+qx+1 and x^3+px^2+qx+1 have a common linear factor, then $(p-1)^2-q(p-1)+1=0$.

33. Prove that one of the roots of $x^3+ax^2+bx+c=0$ is the negative of another if, and only if, $c=ab$.

34. If a polynomial $P(x)$ contains only even powers of x, show that $P(-x)=P(x)$. Any function with this property is called an *even* function. What characteristic has the graph of such a function,

35. If a polynomial $P(x)$ contains only odd powers of x, show that $P(-x)=-P(x)$. Any function with this property is called an *odd* function. What characteristic has the graph of such a function?

***36.** Given $f(x) = x^4 - x^3 - 11x^2 + 9x + 18$, use synthetic division to find $f(2)$, $f(-1)$, $f(-2)$, and $f(-3)$. Applying the factor theorem to these results, determine which of $x-2$, $x+1$, $x+2$, $x+3$ are factors of $f(x)$.

37. Using synthetic division, find the values of

$$x^3 - 3x^2 - 9x + 10$$

for the following values of x: $-3, -2, -1, 0, 1, 2, 3, 4, 5$. Draw the graph of the function.

38. Locate between pairs of consecutive integers the roots of $x^3 - 2x^2 - 12x + 10 = 0$, and compute to three figures the largest root.

39. Show that all the roots of $x^3 + x^2 - 2x - 1 = 0$ are real and compute each correct to two places of decimals.

40. If the three edges of a cube are increased by 1 inch, 2 inches, and 3 inches respectively, the resulting solid has twice the volume of the original cube. Find, to the nearest hundredth of an inch, the edge of the cube.

41. Using the formula of ex. 2, page 121, find the depth to which a cork sphere 4 inches in diameter will sink in water if the specific gravity of the cork is 0.24.

42. One of the famous problems of antiquity was that of duplicating the cube, that is, of constructing a cube with twice the volume of a given cube. A given cube has an edge of 3 inches; find, to the nearest hundredth of an inch, the edge of a cube which has double the volume of this.

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*The remaining exercises under *B* are based on sections 31, 32, and 33 of the text.

C

43. A polynomial is of the fourth degree. Its graph passes through the origin and through $(1,0)$ and is symmetrical about the y-axis. Write the polynomial. How many unknown coefficients does it have? What additional information would suffice to specify the polynomial completely?

44. Show that, if $f(a)=f(b)=f(c)$ where a, b, c are all distinct, the polynomial $f(x)$ leaves a constant remainder when divided by $(x-a)(x-b)(x-c)$.

45. Show that, if a polynomial $f(x)$ leaves a remainder of the form $px+q$ when divided by $(x-a)(x-b)(x-c)$, where a, b, c are all distinct, then $(b-c)f(a)+(c-a)f(b)+(a-b)f(c)=0$.

46. A polynomial $f(x)$ is divided by $(x-a)(x-b)$. Show that the remainder is

$$\frac{f(a)}{a-b}(x-b)+\frac{f(b)}{b-a}(x-a).$$

47. Find the quartic function of x which takes the values $-6, 0, 0, 0, 30$ when $x=-2, -1, 0, 1, 2$, respectively.

48. Given that the cubic equation with real coefficients, $x^3-3x^2-\cdots=0$, has the root $-1+i\sqrt{3}$, find the other roots and complete the equation.

49. Find a condition satisfied by p, q, and r in order that one of the roots of the equation $x^3+px^2+qx+r=0$ may be the average of the other two.

50. Find values of a and b which will make

$$(ax+b)(x^5+1)-(5x+1)$$

divisible by x^2+1.

51. Solve $(x^2+3x+2)(x^2+7x+12)+x^2+5x-6=0$.

52. The function $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ is known to be the square of a linear function of x, y, and z. Show that $af-gh=bg-hf=ch-fg=0$.

53. Find a, b, and c in order that the roots of

$$x^3-15x^2+cx-105=0$$

may be $a-b$, a, and $a+b$.

54. Find a and b in order that the equations

$$x^3+ax^2+11x+6=0 \text{ and } x^3+bx^2+14x+8=0$$

may have two roots in common.

55. If an algebraic equation with *rational* coefficients has a root $a+\sqrt{b}$, where a and b are rational but $\sqrt{b}$ is irrational, show that $a-\sqrt{b}$ is also a root.

56. Solve $x^4-4x^3+4x-1=0$, given that $2-\sqrt{3}$ is a root.

57. Solve $3x^4-25x^3+7x^2+235x-276=0$, given that $5+\sqrt{2}$ is a root.

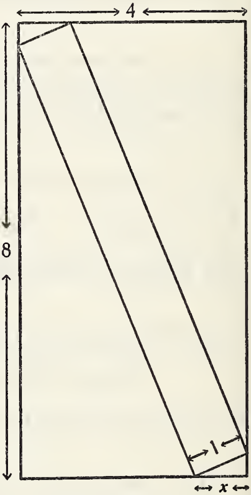
58. The method of synthetic division can be extended to the case in which the divisor is of higher degree than the first. The

division of $2x^4-4x^3+5x^2+8x+3$ by x^2-3x+5 appears as follows:

$$\begin{array}{rrrrrl}
2 & -4 & +5 & +8 & +3 & \underline{|\,3-5} \\
 & 6 & +6 & +3 & & \\
 & & -10 & -10 & -5 & \\
\cline{1-5}
2 & +2 & +1\,| & +1 & -2 &
\end{array}$$

Quotient: $2x^2+2x+1$;

remainder: $x-2$.



Analyse this method and apply it to divide

(*a*) $x^4-6x^3+6x^2-7x-8$ by x^2-2x+4;

(*b*) $2x^4-x^3-6x^2+4x-8$ by x^2-4;

(*c*) x^5+4 by x^2+x+1.

59. A rectangular panel 1 foot wide is to be fitted diagonally on a door 4 ft. by 8 ft. as in the diagram. Show that the distance x satisfies the equation

$$4x^4-16x^3+76x^2+8x-63=0.$$

Calculate the value of x to two significant figures.

CHAPTER V

RATIONAL FUNCTIONS OF ONE OR MORE VARIABLES

RATIO AND PROPORTION

SYSTEMS OF EQUATIONS

34. **Functions of two or more variables.** The functions which have been considered up to this point are functions of a single variable. It is easy to give examples of variable quantities which depend for their values on two or more variables. Thus, to take examples from elementary algebra, the expressions $2x+3y$, $(x+y)^2$, and 2^{x-y} depend for their values on x and y. Again, in geometry, the area of a triangle is given by the formula $\frac{1}{2}bh$, the volume of a cylinder by $\pi r^2 h$, and the hypotenuse of a right triangle by $\sqrt{a^2+b^2}$; in each of these formulas a change in the value of either letter produces a change in the quantity represented. Moreover, in other situations, functions of two or more variables are of common occurrence. The amount of a sum of money at interest depends on the interest rate and the number of interest periods; the heating effect of an electric current flowing through a wire depends on the strength of the current and the resistance of the wire; the weight of a quantity of metal depends on its volume and its density. The definition of a function of two variables is a direct extension of that for one variable.

DEFINITION. *z is a function of x and y if, when a value of x and a value of y are given, a value of z can be determined.*

A function of x and y may be represented by the symbol $f(x, y)$ or $F(x, y)$; other letters are also used. What is essential, in order that there be a function of x and y, is that a rule should be known for determining a value of the function corresponding to a value of x and of y. Just as for functions of one variable (cf. chap. 1), this rule can be given in a variety of ways of which the most useful is by means of an explicit expression. In the following two sections, important types of functions are discussed. It should be noticed that the definitions in all three sections can be extended directly to functions of more than two variables.

35. Polynomials in two variables. A polynomial in two variables is a direct generalization of that in one variable: *it consists of terms which are constant or contain positive integral powers of the variables.* The following are examples:

(*a*) $y^3+2xy^2+4x^2y+5x^3$,

(*b*) $3x^2-2xy+4y^2-6x+5y-7$,

(*c*) x^3+y^3-3xy,

(*d*) $x^2y^3+y^3-3xy^2+2y^2-xy+2x^2-3x+3y+6$,

(*e*) $(2x+3y)^4$.

It will be noticed that, in each of these examples, the terms can be arranged in descending powers of x or of y. Thus, (*a*) is in descending powers of y and (*d*) can be so arranged as follows:

$$y^3(x^2+1)-y^2(3x-2)-y(x-3)+(2x^2-3x+6),$$

the coefficients of the powers of y being polynomials in x. This suggests the following definition.

DEFINITION. *A polynomial in x and y is a function which has or can be made to take the form*

$$A_n(x)y^n+A_{n-1}(x)y^{n-1}+\cdots+A_1(x)y+A_0(x),$$

where the A's are polynomials in x of any degrees.

The terms in this expression can also be arranged in powers of x, in which case the coefficients are polynomials in y.

The term x^2y^3, the first term of (d) above, is said to be of *degree* 5; in general, a term ax^py^q is of *degree* $p+q$ in the variables x and y. If all the terms of a polynomial are of degree n, the polynomial is said to be ***homogeneous of degree n***. Thus, in the foregoing examples, (a) and (e) are homogeneous of degrees 3 and 4 respectively; the other polynomials are *non-homogeneous.*

If a polynomial contains terms of the second degree but none of higher degree, it is a quadratic function. Similar statements apply to cubics and polynomials of higher degrees. The general quadratic function of x and y is conveniently written

$$ax^2+2hxy+by^2+2gx+2fy+c.$$

EXERCISES

1. Which of the following are polynomials in x and y? Justify your answers.

(a) $x^{10}+y^{10}$, (b) $x^{-10}+y^{-10}$, (c) 10^x+10^y,

(d) $(xy-1)^5$, (e) $(xy-1)^{-5}$, (f) $(xy-1)^{5/2}$,

(g) $\dfrac{x^3-y^3}{x-y}$, (h) $\dfrac{x^3-y^3}{x+y}$.

2. Arrange each of the following polynomials in descending powers of x:

(*a*) $x^2y^2+5xy-3y^2-2x^2+7xy^2-x-9y+3$,

(*b*) $(x+2y)^3-3(2x-y)^2+5x-2y-6$, (*c*) $(x-y+1)^3$.

3. Arrange each of the polynominals in ex. 2 in descending powers of y.

4. Write the most general form of

(*a*) a homogeneous quadratic function of x and y;

(*b*) a homogeneous cubic function of x and y;

(*c*) a homogeneous quadratic function of x, y, and z.

5. Write the most general cubic function of x and y.

6. Express each of the following quadratics as the product of two linear factors. (A convenient method is by completing the square of the terms containing x and expressing the quadratic as the difference of two squares.)

(*a*) $x^2+2xy-y^2-4y-2$, (*b*) $x^2-y^2+2x+6y-8$,

(*c*) $x^2+2xy-8y^2+2x+20y-8$.

36. Rational functions. Let us recall that a rational number is any number which can be expressed in the form a/b, where a and b are integers and $b\neq 0$. If we start with any number of integers and combine them by the operations of addition, subtraction, multiplication, and division, the resulting number will be a rational number.

The definition of a rational function is analogous to that of a rational number. If we start with a variable or with more than one variable, *any function which we build up by means of the four operations mentioned is a rational function.* Since such a function can always be reduced to the quotient of two polynomials, we make the following definition.

DEFINITION. *A rational function is one which has or can be made to take the form P/Q, where P and Q are polynomials.*

If the denominator Q is a constant, the rational function reduces to a polynomial; for this reason a polynomial is called an *integral rational function* to distinguish it from the general case of a fractional rational function. As rational functions are analogous to fractions in arithmetic, so polynomials are analogous to integers. In the following examples, the four functions under (*a*) are rational functions of a single variable, while those under (*b*) are rational functions of more than one variable.

$$(a)\quad \frac{t}{t+1},\quad \frac{1}{x^2+1},\quad \frac{(y+3)(2y-1)}{3y+2},\quad \frac{x+\dfrac{x-1}{x+1}}{\dfrac{3}{x}-\dfrac{2}{x+1}};$$

$$(b)\quad \frac{x+y}{x-y},\quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z},\quad \frac{x^2+y^2+z^2}{x+y+z},\quad \frac{x^2-3xy+4y^2}{2x-y}.$$

36.1. An example of a rational function. Let us consider, as an example, the function $y=\dfrac{2}{x^2+1}$, and show the manner of its variation by means of a graph. This can be constructed from the following table of values:

x	0	±1	±2	±3	±4	...
y	2	1	2/5	1/5	2/17	...

From a study of the table of values and of the expression for the function, certain conclusions can be drawn regarding the nature of the function, even before the graph is drawn. These conclusions are as follows:

(*a*) The value of y is positive for any value of x. Hence the graph lies wholly above the x-axis.

(*b*) Since y has the same value when $x=a$ as when $x=-a$, for any value of a, it follows that the graph is symmetrical with respect to the y-axis.

(*c*) As x grows larger and larger, y becomes closer and closer to 0; hence the graph approaches the x-axis more and more closely as the point describing it recedes from the origin.

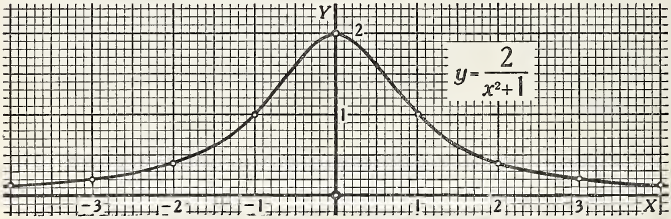


FIG. 20

The graph is drawn in fig. 20. It shows that the function has a maximum value when $x=0$ and that, as x increases from 0, the function continually decreases, having no minimum value.

37. Partial fractions. The problem of adding or subtracting fractions is studied in a first course in algebra. Thus, it is easily verified that

$$(1) \qquad \frac{3}{x+1}+\frac{2}{x-3}=\frac{5x-7}{(x+1)(x-3)}.$$

For some purposes, the left member of (1) furnishes a better means of drawing conclusions regarding the function represented than does the right member. It is then an advantage, when given the latter expression, to be able to separate it into the two former fractions, which are called its ***partial fractions.***

To arrive at a way of doing this, we shall assume that the composite fraction is the sum of two partial fractions whose denominators are the factors of the given denominator and whose numerators are to be found. Thus,

$$\frac{5x-7}{(x+1)(x-3)}=\frac{a}{x+1}+\frac{b}{x-3}. \tag{2}$$

By multiplying both members of this equation by $(x+1)(x-3)$, we obtain

$$5x-7=a(x-3)+b(x+1)$$

or

$$5x-7=(a+b)x-3a+b. \tag{3}$$

Since equation (2) is to be true for all values of x, except $x=-1$ and $x=3$ (for which the given fraction has no meaning), therefore (3) also holds for the same values of x. But, if two linear functions are equal for more than one value of x, the coefficients of like powers of x in the two functions are equal (sec. 28.3, cor. 2). Hence, from (3),

$$a+b=5,$$

$$-3a+b=-7.$$

These equations give $a=3$, $b=2$, so that the partial fractions are those of (1).

EXERCISES

1. Which of the following are rational functions of x and which are not? Justify your answers.

(*a*) $2x^2-5+3x^{-2}$, (*b*) $\frac{1}{5x-4}$,

(*c*) $x+2\sqrt{x}+1$, (*d*) $x^{10}+1/x^{10}$,

(*e*) $2^{2x}+3\cdot 2^x+1$, (*f*) $(\sqrt[3]{x}+1)^3$,

(*g*) $(x^4+a^4)/(x^3+a^3)$, (*h*) $(\sqrt{x}+2)/(\sqrt{x}-2)$.

2. Of the following, which are rational functions of x and y? Give your reasons.

(a) x^2+xy+y^2, (b) $x^{3/2}-x^{1/2}y^{1/2}+y^{3/2}$,

(c) $\dfrac{x+y}{x-y}$, (d) $\dfrac{x+\sqrt{y}}{x-\sqrt{y}}$,

(e) $(x/y)^3-2(x/y)^2+5(x/y)-6$, (f) $(1/x+1/y)^{50}$,

(g) $a^{x+y}+b^{x-y}$, (h) $\log x^2y$.

3. Express each of the following rational functions as the quotient of two polynomials:

(a) $4-\dfrac{2}{x+2}+\dfrac{3}{(x+2)^2}$, (b) $\dfrac{\dfrac{x-2}{x+2}+\dfrac{x+2}{x-2}}{\dfrac{1}{x+2}+\dfrac{1}{x-2}}$, (c) $\dfrac{3}{x+\dfrac{4}{x-\dfrac{2}{x}}}$.

4. If, in the graph of $\dfrac{2}{x^2+1}$ (fig. 20), the unit in terms of which x and y are measured were 1 mile, show that, when $x=250$ miles, the curve lies approximately two inches from the x-axis.

5. From fig. 20, estimate the value of x at which $\dfrac{2}{x^2+1}$ decreases most rapidly as x increases.

6. Draw the graph of $1/x^2$. Does the graph cross either axis? Has it an axis of symmetry? For what range of values of x does the function increase with x? For what values does it decrease as x increases? Has the function a maximum? Has it a minimum? For what values of x does it change rapidly? For what values, slowly?

7. Separate into partial fractions:

$$(a)\ \frac{3x-5}{(x-1)(x-2)},\quad (b)\ \frac{2x}{x^2-1},\quad (c)\ \frac{x+3}{x^2+x}.$$

8. Separate into partial fractions:

$$(a)\ \frac{x+15}{x^2-9},\quad (b)\ \frac{x+12}{2x^2+3x-2}.$$

9. Which of the following statements are always true and which are not? For each of those which you assert to be false, give an example to prove your assertion. (*a*) The sum of two rational functions is rational. (*b*) The product of two rational functions is rational. (*c*) The square root of a rational function is rational. (*d*) If y is a rational function of x, then x is a rational function of y. (*e*) If y is an irrational function of x, then x is an irrational function of y.

10. If, in the equation of a graph, a change in the sign of x does not alter the equation, show that the graph is symmetrical about the y-axis. State a similar test for symmetry about the x-axis.

11. If, in the equation of a graph, changing the signs of both x and y does not alter the equation, show that the graph is symmetrical about the origin though it need not be symmetrical about either axis.

12. Of the following equations, which have graphs symmetrical about (*a*) the x-axis, (*b*) the y-axis, (*c*) the origin?

$$\text{(i) } y=x^2+2,\quad \text{(ii) } y^2=x+4,\quad \text{(iii) } 2x^2+3y^2=12,$$

$$\text{(iv) } x/y+y/x=1,\quad \text{(v) } xy=6,\quad \text{(vi) } x^4+y^4=48.$$

13. Make up an example of your own for each of the following: (*a*) an equation whose graph is symmetrical about the x-axis but not about the y-axis, (*b*) an equation whose graph

is symmetrical about both x and y axes, (c) an equation whose graph is symmetrical about the origin but not about either axis.

38. Ratio and proportion. The fraction a/b is also called the ***ratio*** of the two numbers a and b, or the ratio of a to b; it is sometimes written in the alternative form $a : b$. Thus, a rational function $\frac{P(x)}{Q(x)}$ is the ratio of the numerator $P(x)$ to the denominator $Q(x)$. The numerator and denominator of a ratio are sometimes called its ***terms***. The notion of ratio is commonly extended to apply to two concrete magnitudes of the same kind, such as two lengths, two volumes, two forces, etc. The ratio required is then the ratio of the measures of the two magnitudes when these are expressed in the same units; for example, the ratio of 3 feet to 18 inches is 36:18 or 2:1. Ratios are of much use in geometry and trigonometry as well as in algebra.

When two ratios are equal, the resulting equation is called a ***proportion*** and the four numbers involved are said to be ***in proportion***. Thus $\frac{2}{5}=\frac{12}{30}$ is a proportion and $\frac{y}{b}=\frac{x}{a}$ is a proportion, when the numbers represented satisfy this equation. If two variables change in such a manner that their ratio remains constant, then one of these variables is said to be ***proportional to*** the other;* that is, if $y/x=m$, a constant, then y is proportional to x. This use of the term is illustrated in the following statements: the distance travelled by a car at constant speed is proportional to the time taken; the weight of a quantity of iron is proportional to its volume; the amount by which a coiled spring is stretched is (within limits) proportional to the stretching force. Other examples will occur to the reader.

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*Cf. sec. 8, page 32.

When two ratios are equal, many useful results can be deduced from the resulting proportion. Thus, if $\frac{x}{a}=\frac{y}{b}$, it follows that $bx=ay$, also that $\frac{b}{y}=\frac{a}{x}$, and that $\frac{x}{y}=\frac{a}{b}$. Any one of the numerators and denominators (supposed different from zero) in a proportion can be expressed in terms of the other three. Thus, $x=\frac{ay}{b}$, $a=\frac{bx}{y}$, etc. The equation $\frac{a}{x}=\frac{x}{b}$ is a particular proportion which yields $x^2=ab$. If a and b are numbers of the same sign, then $\sqrt{ab}$ and $-\sqrt{ab}$ are called ***mean proportionals*** between a and b; of these mean proportionals, the more useful one is that which has the same sign as a and b. Results of a more extended sort can be deduced from a proportion by either of two methods which are illustrated in the following examples. All results are, of course, subject to the condition that no denominator vanishes.

EXAMPLE 1. Given $\frac{x}{a}=\frac{y}{b}$, show that each fraction $=\frac{x+y}{a+b}$.

First method:

Expressing x in terms of the other terms of the proportion, we have $x=\frac{ay}{b}$. Then

$$\frac{x+y}{a+b}=\frac{\frac{ay}{b}+y}{a+b}=\frac{y\left(\frac{a}{b}+1\right)}{a+b}=\frac{y}{b},$$

which is also equal to $\frac{x}{a}$.

Second method:

Since the given fractions are equal, we may denote the value of each of them by k. Then $x = ak$, $y = bk$. Hence

$$\frac{x+y}{a+b}=\frac{ak+bk}{a+b}=\frac{k(a+b)}{a+b}=k,$$

which is the value of $\frac{x}{a}$ and of $\frac{y}{b}$.

Example 2. Given $\frac{x}{a}=\frac{y}{b}$, show that $\frac{x+a}{x-a}=\frac{y+b}{y-b}$.

First method:

We have $x=\frac{ay}{b}$. Then

$$\frac{x+a}{x-a}=\frac{\frac{ay}{b}+a}{\frac{ay}{b}-a}=\frac{a(y+b)}{a(y-b)}=\frac{y+b}{y-b},$$

which was to be proved.

Second method:

Denote each of the given fractions by k. Then $x=ak$, $y=bk$.

Therefore $$\frac{x+a}{x-a}=\frac{ak+a}{ak-a}=\frac{k+1}{k-1}.$$

Also $$\frac{y+b}{y-b}=\frac{bk+b}{bk-b}=\frac{k+1}{k-1}.$$

Hence $$\frac{x+a}{x-a}=\frac{y+b}{y-b}$$

since each is equal to $\frac{k+1}{k-1}$.

EXERCISES

1. Give the ratio of (*a*) 4 metres to 75 centimetres, (*b*) 700 pounds to 2 tons, (*c*) 7 gallons to 24 pints.

2. Divide $330 into two shares in the ratio of 7 to 11. (Hint: Let the shares be $7x$ and $11x$.)

3. (*a*) If from one end of a line 2/5 of its length is cut off, in what ratio does the point of section divide the line?

(*b*) If a line is divided in the ratio $a:b$, what fraction of the line is each part?

4. (*a*) Express a velocity of 40 miles per hour in feet per second. (*b*) Express an acceleration of 980 cm. per sec. per sec. in metres per min. per min.

5. Find a mean proportional between

(*a*) 4 and 16, (*b*) 6 and 24, (*c*) $\dfrac{r+s}{r-s}$ and r^2-s^2.

6. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, show that each fraction $=\dfrac{x+y+z}{a+b+c}$.

Express this theorem in words.

7. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, show that each fraction $=\dfrac{lx+my+nz}{la+mb+nc}$.

8. If $\dfrac{x}{a}=\dfrac{y}{b}$, show that $\dfrac{lx+ma}{lx-ma}=\dfrac{ly+mb}{ly-mb}$.

9. If $\dfrac{x}{a}=\dfrac{y}{b}$, find other ratios, one for each, which are equal to the following:

$$\frac{x+a}{x},\quad \frac{a-x}{a},\quad \frac{3x-4a}{5x+2a},\quad \frac{\sqrt{x^2+a^2}}{x+a}.$$

10. If $\frac{x}{a}$, $\frac{y}{b}$, and $\frac{z}{c}$ are unequal fractions in which all the letters denote positive numbers, show that $\frac{x+y+z}{a+b+c}$ lies in value between the greatest and the least of the given fractions.

11. Find three numbers in the ratio of 2:3:4 such that their sum is 225.

12. What number, added to each term of 3/7, will make the ratio become 2/3?

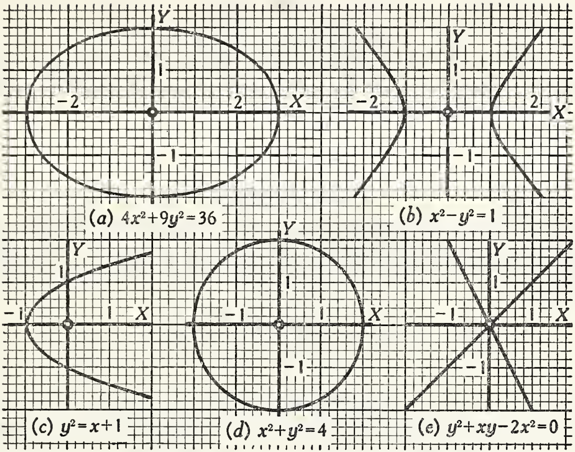
13. If $2x+5y=0$, find the ratio $\frac{x}{y}$, and show that it is equal to the ratio $\frac{x-5}{y+2}$.

39. Systems of equations. Consider an equation of the form $P(x, y)=0$, where $P(x, y)$ is a polynomial in x and y. If a constant value x_0 is assigned to x, the equation becomes $P(x_0, y)=0$, which is an algebraic equation in y. This algebraic equation has certain roots and hence there exist values of y corresponding to $x=x_0$. By assigning different values to x_0, we can build up a table of values of x and y and from it obtain a graph or *locus* corresponding to $P(x, y)=0$. This graph and the equation $P(x, y)=0$ are related by the facts that the coordinates of any point on the graph satisfy the equation and, conversely, that any pair of values of x and y which satisfy the equation are the coordinates of a point on the graph. As an example, the equation

$$4x^2+9y^2-36=0$$

leads to the following table of values:

x	-3	-2	-1	0	1	2	3
y	0	±1.5	±1.9	±2	±1.9	±1.5	0



(a) $4x^2+9y^2=36$

(b) $x^2-y^2=1$

(c) $y^2=x+1$

(d) $x^2+y^2=4$

(e) $y^2+xy-2x^2=0$

FIG. 21

The graph of this equation is shown in fig. 21, along with some other examples of loci given by equations of the second degree in x and y.

Consider now a system of two such equations,

$$P(x, y) = 0$$

$$Q(x, y) = 0,$$

each of which represents some locus. If these loci intersect in a point (x_1, y_1), then x_1 and y_1 together satisfy both equations. Hence *the geometric problem of finding the intersections of a pair of curves is only another aspect of the algebraic problem of solving a pair of equations in two unknowns.*

If the equations are of the first degree,

$$ax+by+c = 0$$

$$a_1x+b_1y+c_1 = 0,$$

the corresponding loci are straight lines. These two lines can be related to each other in three possible ways: (i) the lines may intersect, (ii) the lines may coincide, (iii) the lines may be parallel. In case (i), the two equations have a *unique* solution. This is illustrated by the equations

$$2x+3y=4$$
$$3x+4y=7,$$

which have the unique solution, $x=5$, $y=-2$. In case (ii), the equations have an unlimited number of solutions. Thus, the equations

$$2x+3y=4$$
$$4x+6y=8$$

represent the same straight line and are satisfied by the coordinates of every point on this line. In case (iii), the equations have no solution and are said to be *inconsistent.* Thus

$$2x+3y=4$$
$$4x+6y=9,$$

which represent parallel lines, are seen to be inconsistent; they cannot both hold for any values of x and y.

39.1 Linear homogeneous equations. The linear equation, $ax+by+c=0$, does not determine unique values of x and y. If however, $c=0$, the *ratio* of the unknowns is uniquely determined. For from $ax+by=0$, it follows that $\frac{x}{y}=-\frac{b}{a}$ if $a\neq 0$, and $\frac{y}{x}=-\frac{a}{b}$ if $b\neq 0$. Since each term in this latter equation is of the first degree in x and y, it is called a *homogeneous* linear equation. Similarly, from two homogeneous linear equations in three unknowns, it is, in general, possible to determine uniquely the ratios of the unknowns. Thus the equations

$$ax+by+cz=0$$
$$a_1x+b_1y+c_1z=0$$

can be written

$$a\frac{x}{z}+b\frac{y}{z}+c=0$$

$$a_1\frac{x}{z}+b_1\frac{y}{z}+c_1=0$$

After multiplying these equations respectively by b_1 and b, we can eliminate $\frac{y}{z}$ by subtraction and obtain

$$\frac{x}{z}(ab_1-a_1b)+cb_1-c_1b=0$$

or

$$\frac{x}{z}=\frac{bc_1-b_1c}{ab_1-a_1b}.$$

Similarly,

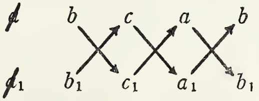
$$\frac{y}{z}=\frac{ca_1-c_1a}{ab_1-a_1b}.$$

These results can be written in the convenient form

$$\frac{x}{bc_1-b_1c}=\frac{y}{ca_1-c_1a}=\frac{z}{ab_1-a_1b}.$$

The usefulness of this method of expressing the ratios of the unknowns, in a pair of homogeneous linear equations in three unknowns, justifies our calling attention to the following rule for obtaining the denominators.

(*a*) Write the coefficients of *x*, *y*, and *z*, repeating those of *x* and *y*, thus:

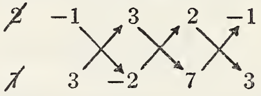


(*b*) Stroke off the first pair and associate the remaining letters in the manner indicated by the arrows.

Example 1.

$$2x - y + 3z = 0$$
$$7x + 3y - 2z = 0.$$

The suggested scheme of numbers is, in this case,



whence

$$\frac{x}{2-9} = \frac{y}{21+4} = \frac{z}{6+7},$$

or

$$\frac{x}{-7} = \frac{y}{25} = \frac{z}{13}.$$

After experience with a few problems, the student may be able to dispense with writing the scheme of numbers and to form the denominators mentally.

39.2. A system of one linear and one quadratic equation. The solution of such a pair of equations gives, geometrically, the intersections of a straight line with a curve whose equation is of the second degree. An inspection of the curves of the

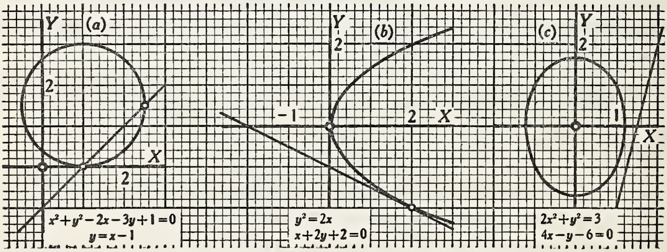


Fig. 22

second degree, shown in figures 21 and 22, leads to the conclusion that a straight line can be related to such a curve in three principal ways: (i) it may cut the curve in two distinct points as in fig. 22a, (ii) it may touch the curve, as in fig. 22b, (iii) it may have no point in common with the curve, as in fig. 22c. Corresponding to these three cases, we may expect, in the algebraic solution, to obtain (i) two distinct pairs of real roots for x and y, (ii) one pair of real roots or, otherwise expressed, two equal pairs of real roots, (iii) two pairs of imaginary roots. The algebraic solution can be carried out as in the following example, for which the loci represented are those shown in fig. 22a.

EXAMPLE 2. Solve the system:

$$x^2+y^2-2x-3y+1=0$$
$$y=x-1.$$

Since the first equation is to be true for the same values of x and y as the second, we can substitute $y=x-1$ in the first.

Hence,

$$x^2+(x-1)^2-2x-3(x-1)+1=0,$$

which gives

$$2x^2-7x+5=0,$$

or

$$(x-1)(2x-5)=0,$$

whence $x=1$ or $5/2$. When $x=1$, $y=0$, and when $x=5/2$, $y=3/2$. Hence the loci represented by the given equations intersect in the points $(1, 0)$ and $(5/2, 3/2)$.

39.3. Systems of two quadratic equations. For a system of two quadratic equations in x and y, the method of solution which suggests itself most readily is that of solving one of these equations for y (or x) and substituting the resulting expression in the other equation. The result of this substitution is, in general, an equation of the fourth degree. Thus, given

$$\begin{aligned} x^2+y^2-3y-2 &= 0 \\ x^2-y^2+2x &= 0, \end{aligned}$$

we find, from the second equation, $y=\pm\sqrt{x^2+2x}$. When this value of y is substituted in the first equation, we have

$$2x^2+2x\pm3\sqrt{x^2+2x}-2=0,$$

or

$$2(x^2+x-1)=\pm3\sqrt{x^2+2x},$$

which, on squaring both sides and reducing, gives

$$4x^4+8x^3-13x^2-26x+4=0.$$

As we have developed no general methods of solving quartic equations, it will be only in special cases that we shall be able to carry through the solution of a pair of quadratic equations in two unknowns. The fact that the equation in x is of the fourth degree and has real coefficients shows that it is satisfied by four values of x, of which all may be real, two real and two imaginary, or all imaginary. Also, various special cases arise, in which two real roots are equal, or four real roots are equal in pairs,

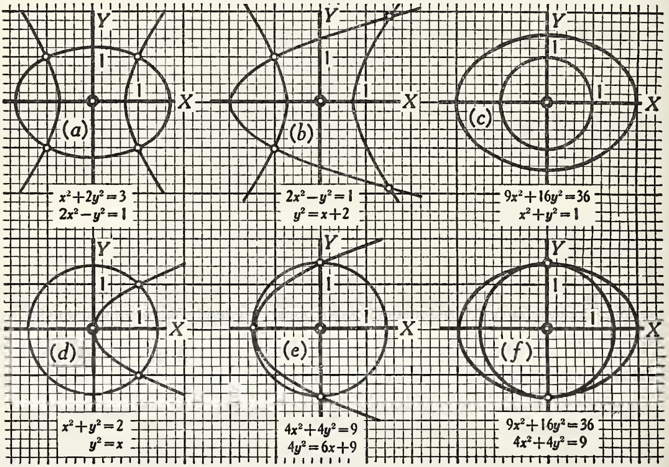


FIG. 23

etc. Similar statements apply to the four values of y which correspond to those of x. In fig. 23 are shown the graphs of six pairs of quadratic equations. Of these, the equations represented in (*a*) and (*b*) have 4 distinct pairs of real roots, those in (*c*) have 4 pairs of imaginary roots, those in (*d*) have two pairs of real and two of imaginary roots, those in (*e*) have a pair of coincident and a pair of distinct real roots, and those in (*f*) have two pairs of coincident real roots.

In some cases, the quartic equation in x, which results from the pair of equations by elimination of y, is easily solvable. If, for example,

$$x^2+y^2 = 5$$
$$xy = 2,$$

then $y = \dfrac{2}{x}$ and $x^2 + \dfrac{4}{x^2} = 5$, giving $x^4-5x^2+4=0$. The roots of this quartic are easily found, since $(x^2-4)(x^2-1)=0$, and hence the pair of equations can be solved.

39.4. The following examples illustrate certain ingenious devices which are useful for equations of the types represented.

EXAMPLE 3. Solve the system: $4x^2+y^2=41$
$$xy = 10.$$

From the second equation, $4xy=40$; combining this with the first equation, we get

$$4x^2+4xy+y^2=81 \text{ and } 4x^2-4xy+y^2=1,$$

or $$(2x+y)^2=81 \text{ and } (2x-y)^2=1.$$

Hence $$2x+y=\pm 9$$

and $$2x-y=\pm 1.$$

It follows that $x = 5/2, 2, -5/2$ or -2, and, since $y = 10/x$, the corresponding values of y are 4, 5, -4, and -5. The loci represented by the equations intersect in the four distinct points.

$$(5/2, 4),\ (2, 5), (-5/2, -4), \text{ and } (-2, -5).$$

EXAMPLE 4. Solve the system:
$$x^2-3xy+2y^2=4$$
$$2x^2-2xy-5y^2=11.$$

The peculiarity of these equations is that the left members are homogeneous of the second degree and the right members are constants. By multiplying these equations through by 11 and 4 respectively and subtracting, we obtain the homogeneous equation

$$3x^2-25xy+42y^2=0,$$

or
$$(3x-7y)(x-6y)=0,$$

whence $x=\frac{7y}{3}$ or $6y$. The solution of the given system is now obtained by substituting these expressions for x in the first equation, as follows:

If $\quad x=\frac{7y}{3},$

then $\frac{49}{9}y^2-7y^2+2y^2=4,$

or $\quad y^2=9$

and $\quad y=\pm 3.$

Hence pairs of corresponding values of x and y are

$$x=7, \quad y=3,$$

and $\quad x=-7, \; y=-3.$

If $\quad x=6y,$

then $\quad 36y^2-18y^2+2y^2=4,$

or $\quad y^2=\frac{1}{5}$

and $\quad y=\pm\frac{\sqrt{5}}{5}.$

Pairs of corresponding values of x and y are

$$x=\frac{6\sqrt{5}}{5}, \; y=\frac{\sqrt{5}}{5},$$

and $x=\frac{-6\sqrt{5}}{5}, \; y=\frac{-\sqrt{5}}{5}.$

Example 5. Solve the system:

$$x+y-z=0$$
$$3x-3y+z=0$$
$$x^2+y^2+z^2=14.$$

From the first two equations, which are homogeneous, we find the ratios of x, y, and z:

$$\frac{x}{1-3}=\frac{y}{-3-1}=\frac{z}{-3-3},$$

or

$$\frac{x}{-2}=\frac{y}{-4}=\frac{z}{-6},$$

or

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}.$$

Denoting each of these fractions by k and substituting $x=k$, $y=2k$, $z=3k$ in the third equation, we get

$$k^2+4k^2+9k^2=14,$$

whence $k=\pm 1$. The value $k=1$ gives the solution $x=1$, $y=2$, $z=3$, and $k=-1$ gives $x=-1$, $y=-2$, $z=-3$.

EXERCISES

1. Write an equation which is inconsistent with $7x-3y=10$, and one inconsistent with $x^2+y^2=10$.

2. Determine the ratios of $x:y:z$ in the following pairs of homogeneous linear equations:

(*a*) $x-2y+z=0$
$3x+2y-3z=0,$

(*b*) $3x+4y-3z=0$
$8x-y-z=0,$

(*c*) $x-my+2mz=0$
$mx+2y-3z=0.$

Solve the following systems of equations:

3. $x+y=11$
$xy=24.$

4. $x^2+10xy-y^2=100$
$x^2+y^2=20.$

5. $x^2+y^2-5x-5y-2=0$
$x^2-5xy-6y^2=0.$

6. $xy+5x-8y=8$
$2xy-7x+11y=29.$

7. $x^2-xy+y^2=21$
$x^3+y^3=63.$
(Hint: divide the members of the second equation by those of the first.)

8. $x^2+y^2-10x=0.$
$2x-y-20=0.$

9. $2x^2-5xy+3y^2=1$
$3x^2-5xy+2y^2=4.$

10. $3x-5y+z=0$
$x-y-z=0$
$xy+yz+zx=11.$

11. $x+2y+z=0$
$-x+2y+2z=0$
$x^2+y^2+z^2=29.$

12. The perimeter of a rectangular field measures 340 rods and its diagonal, 130 rods. Find the length and width of the field.

13. Three numbers are such that the sum of the second and third is 6 times the first, the difference between the third and second is twice the first, and the sum of the cubes of the numbers is 73. Find the numbers.

EXERCISES ON CHAPTER V

A

1. What is the ratio of (*a*) 60 miles per hour to 66 feet per second, (*b*) 30 cubic feet to 5 cubic yards?

2. Find a mean proportional between (*a*) $\frac{3}{4}$ and 48, (*b*) a^4b^2 and 1.

3. Find x in each of the following proportions:

(*a*) $-4:5=6:x$, (*b*) $p^2q^2:x=p^2:q^2.$

4. Solve the system:

$$2x^2-xy=10$$
$$3x^2-y^2=11.$$

5. Find the ratios $x:y:z$, given that

$$5x-y-z=0 \text{ and } 3x-3y+5z=0.$$

6. If $x/3=y/4=z/6$, show that each fraction equals $x-2y+z$.

7. If $x/2=y/3$, write a function of y which is equal to (*a*) $(x+2)/(x-2)$, (*b*) $(x^2+4)/(x^2-4)$, (*c*) $(ax+2b)/2$.

8. If $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{4}$, evaluate $\dfrac{x-3y+4z}{3x+y-2z}$.

9. A log 35 feet long is divided into two parts whose lengths are as 2 is to 3. Find the lengths of the parts.

10. If $\dfrac{m}{n}=\dfrac{n}{p}$, prove that $\dfrac{n+p}{p}=\dfrac{m-p}{n-p}$.

11. Solve the system of equations represented in fig. 22b.

12. The perimeter of an isosceles triangle is 14 inches and its altitude is 5 inches. Find the lengths of its sides.

13. For each of the following equations make a table of values of x and y. Plot on one figure the graphs of the two equations

$$(a)\ y^2-x^2=0, \quad (b)\ y^2-x^2=4.$$

B

14. A certain ellipse has the equation $4x^2+9y^2=25$. Show that the straight line $8x+9y=25$ meets this ellipse in only one point and is, therefore, a tangent to it. Find the point of tangency.

15. Draw the graph of $y=\frac{x}{1+x^2}$. Has the graph symmetry about (*a*) a line, (*b*) a point? Has the function a maximum? Has it a minimum? For what value of x does y appear to increase most rapidly as x increases? Estimate the values of x for which y decreases most rapidly as x increases.

16. Solve the systems of equations represented in figures 23(b) and 23(e), and check your solutions by means of the figures.

17. Solve the system:

$$\begin{aligned} xy+x-y&=-17\\ x^2+y^2&=65. \end{aligned}$$

(Hint: Set $x-y=m$ and $xy=n$; then $x^2+y^2=m^2+2n$. Solve first for m and n.)

18. Solve the system:

$$\begin{aligned} x^2+y^2+3x+3y-28&=0\\ xy-\ 6&=0. \end{aligned}$$

19. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that each of these ratios is equal to

$$\left\{\frac{3x^3-2y^3+5z^3}{3a^3-2b^3+5c^3}\right\}^{1/3}.$$

20. If a, b, c, and d are in proportion, show that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

21. Solve the system:

$$8x+2y+7z=3x-5y-6z=0,\quad x^2+y^2-2z^2=8.$$

22. Resolve into partial fractions

$$(a)\ \frac{5x}{x^2+x-6},\qquad (b)\ \frac{x-12}{(2x+1)(x-2)}.$$

23. An open box is made from a rectangular sheet of cardboard 1000 square centimetres in area. This is done by cutting a 5-centimetre square from each corner, bending up the flaps, and fastening the edges together. If the volume of the box so obtained is 2000 cubic centimetres, find the length and width of the original sheet of cardboard.

24. Find the values of m for which the following system has only one solution:

$$\begin{aligned} x^2+y^2-6x-6y+10&=0 \\ x+y+m&=0. \end{aligned}$$

25. Solve the system:

$$\begin{aligned} 3(x^2+y^2)-2xy&=27 \\ 4(x^2+y^2)-6xy&=16. \end{aligned}$$

26. A rectangular field is 10 rods long and 5 rods wide. By what amounts must the length and width be changed in order that both its area and its perimeter may be doubled? (Give answers correct to hundredths of a rod.)

27. (*a*) In the ratio a/b, it is given that a and b are positive numbers and $b>a$. Prove that the result of adding a positive number p to both numerator and denominator is to give a ratio nearer in value to 1 than the original ratio.

(*b*) If, in part (*a*), $b<a$, show that the theorem is still true.

28. If a and b are positive numbers, what is the effect on a/b of subtracting the positive number p from both numerator and denominator, given that p is less than either a or b?

29. If $x>y>0$, arrange in order of increasing magnitude the fractions: x/y, $(x+1)/(y+1)$, $(x+2)/(y+2)$.

30. If $0<x<y$, arrange in order of increasing magnitude: x/y, $(x-1)/(y-1)$, $(x+1)/(y+1)$.

31. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\cdots$, prove that each of these ratios is equal to

$$\left\{\frac{px^m+qy^m+rz^m+\cdots}{pa^m+qb^m+rc^m+\cdots}\right\}^{1/m}.$$

32. Show the equivalence of the following statements: y is proportional to x; y varies directly as x; y is a homogeneous linear function of x; the graph traced by the point (x, y) is a straight line through the origin.

33. Given the equation of a graph, show how you would obtain the equation of its mirror image in (*a*) the x-axis, (*b*) the y-axis. Write the equations of the mirror images in the two axes of (i) $3x+4y=5$, (ii) $x^2=x+y$, (iii) $xy=x+1$.

34. By observing how increases in x affect y, trace the graph of $y=1/(1-x)$. Describe the change in y as x increases from -10 to 10. Is it possible to draw this part of the graph without removing your pencil from the paper?

35. For the function $y=1/(1-x^2)$, find what values of x make y (*a*) positive, (*b*) negative, (*c*) very large numerically, (*d*) very small numerically. From the information in your answers trace the graph of the function.

C

36. Solve the system:

$$x^2+4y^2-15x=10(3y-8)$$
$$xy=6.$$

37. Find for what values of r the circle $x^2+y^2=r^2$ meets the ellipse $4x^2+9y^2=36$ in (*a*) no real point, (*b*) two real points, (*c*) four real points.

38. If a, b, c are real and not all equal, find the condition which they must satisfy in order that

$$ax+by+c=0$$
$$bx+cy+a=0$$
$$cx+ay+b=0$$

may be a consistent set of equations. Assuming that the condition is satisfied, solve for x and y.

39. Given that

$$x^2+y^2+z^2=1,\ a^2+b^2+c^2=1, \text{ and } ax+by+cz=1,$$

the letters representing real numbers, prove that

$$x=a,\ y=b, \text{ and } z=c.$$

40. Find four numbers which satisfy the following four conditions:

(*a*) the ratio of the first to the second equals the ratio of the third to the fourth ;

(*b*) the sum of the first and fourth is 35 ;

(*c*) the sum of the second and third is 30 ;

(*d*) the sum of the squares of the numbers is 1261.

41. Solve the system:

$$\begin{aligned} x^3+y^3-z^3&=86-3xyz\\ x^2+y^2-z^2&=8-2xy\\ x+y-z&=2. \end{aligned}$$

42. Solve the system:

$$\begin{aligned} xy+5(x+y)&=47\\ yz+5(y+z)&=65\\ zx+5(z+x)&=55. \end{aligned}$$

CHAPTER VI

SERIES OF NUMBERS. THE PROGRESSIONS

40. **Sequences of numbers.** If, in a function $f(n)$, the positive integers are substituted in succession for n, there is obtained a *sequence* of numbers,

$$f(1), f(2), f(3), \cdots .$$

If, for example, $f(n)=2n+5$, then

$$f(1)=2\times1+5=7,\ f(2)=2\times2+5=9,\ f(3)=2\times3+5=11$$

and so on, the resulting sequence being

$$7,\ 9,\ 11,\ 13,\ \cdots .$$

Again, if $f(n)=3\times2^n$, then

$$f(1)=3\times2=6,\ f(2)=3\times2^2=12,\ f(3)=3\times2^3=24$$

and so on, and the resulting sequence is

$$6,\ 12,\ 24,\ 48,\ \cdots .$$

The dots at the end indicate that the numbers or terms of the sequence may be continued as far as desired. We may, for example, have a series of 20 terms, one of 36 terms, etc.

The student should verify, by substitution, that the sequences corresponding to the functions n, $3n-1$, n^2, 3^n are respectively

$$\begin{array}{l} 1,\ 2,\ 3,\ 4, \cdots \\ 2,\ 5,\ 8,\ 11, \cdots \\ 1,\ 4,\ 9,\ 16, \cdots \\ 3,\ 9,\ 27,\ 81, \cdots . \end{array}$$

DEFINITION. *A* **sequence** *of numbers is a set of numbers arranged in a definite order and formed according to a definite law. The successive numbers are called the terms of the sequence.*

It is often convenient to denote the terms of a sequence by $t_1, t_2, t_3, \cdots$; thus t_{20} means the 20th term and t_n, the nth term. The sequence is then denoted by

$$t_1,\ t_2,\ t_3,\ \cdots$$

Closely related to a sequence is a series, in which the terms are connected by + signs, thus:

$$t_1+t_2+t_3+\ \cdots\ .$$

In studying a series we are generally concerned with a definite number of terms; hence we speak of the first term, the last term, and the number of terms of the series. The feature of most importance in a series is the law of formation of the successive terms. Consider, for example, the second of the foregoing series,

$$2,\ 5,\ 8,\ 11,\ \cdots$$

which was formed from the function $3n-1$ by giving to n the values 1, 2, 3, $\cdots$. Since $3n-1$ is the nth term, we have, for this series,

$$t_n=3n-1.$$

When the nth term is known, the successive terms can be formed by substitution in it; hence the nth term gives the law by which the terms are formed. On the other hand, this law may be expressed in words without explicit use of the nth term. Thus, in the series just mentioned, each term is formed by adding 3 to the preceding term, and this is a statement of the law of the series. By means of this fact, the series can be continued as far as desired.

A consideration of the law of a series suggests two converse problems. One is that of building up the series when the nth term is given. The second is that of detecting the law of the series when the first few terms are given. The first problem has already been illustrated. As an example of the second, consider the series

$$12+8+4+0+\cdots.$$

Each term is formed by adding -4 to the preceding term, so that the 5th, 6th, and 7th terms are -4, -8, and -12, respectively.

Again, in the series

$$3+6+12+24+\cdots$$

the law is that each term is formed by multiplying the preceding term by 2. In the series

$$1+3+6+10+\cdots$$

the law is not so obvious until one notices that the second term is 2 more than the first, the third is 3 more than the second, and the fourth is 4 more than the third. To preserve this law, we take for the fifth term 5 more than the fourth, etc.; hence the 5th, 6th, and 7th terms are 15, 21, and 28, respectively. As a last example, the series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$$

has for its fifth term $\frac{5}{6}$, for its tenth term $\frac{10}{11}$, and for its nth term $\frac{n}{n+1}$.

EXERCISES

1. Continue each of the following series three terms further so as to preserve the law of the series:

(*a*) $1+3+5+7+\cdots;$

(*b*) $1+3+9+27+\cdots;$

(*c*) $1^2+2^3+3^4+\cdots;$

(*d*) $2+5+10+17+\cdots;$

(*e*) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots;$

(*f*) $1+2x+3x^2+4x^3+\cdots.$

2. Write the first four terms of the series for which the nth term is

(*a*) $3n+4$, (*b*) $n+\frac{1}{n}$, (*c*) $\log\frac{n+1}{n}$, (*d*) $3(2^n)$, (*e*) n^{-2}.

3. Write the first four terms in each of the series described as follows:

(*a*) the first term is 1 and each term is 6 more than 3 times the preceding term;

(*b*) the terms are the reciprocals of the successive odd integers, starting with 1;

(*c*) the terms are the cubes of the positive integers;

(*d*) the terms are the successive positive integral powers of 3.

41. The arithmetic progression. *A series in which each term, after the first, is obtained by adding the same number to the preceding term is called an arithmetic series or an* ***arithmetic progression*****. The number which is added each time is called the*

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*Sometimes the abbreviation A.P. is used for "arithmetic progression".

common difference; hence, the common difference is obtained by subtracting any term from the one following it.

EXAMPLES:

The series 4, 7, 10, 13 is an arithmetic progression whose common difference is 3.

The series $a, a-b, a-2b, a-3b$ is an arithmetic progression whose common difference is $-b$.

It is seen that any arithmetic series may be represented by

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots \tag{1}$$

in which a stands for the first term, and d for the common difference.

42. The *n*th term of an arithmetic progression. By inspecting the general arithmetic progression (1), we observe that *the coefficient of d in each term is one less than the number of the term.* Since this law holds for all terms, the nth term of an arithmetic progression is given by the formula

$$\boxed{t_n = a + (n-1)d.} \tag{2}$$

Sometimes the nth term is denoted by l, especially when it is the last term of the series. Hence,

$$\boxed{l = a + (n-1)d.} \tag{3}$$

42.1. The nature of the nth term as a function of n is apparent upon examination of formula (3) which gives

$$l = a+(n-1)d = nd+a-d.$$

Since, in a given arithmetic series, a and d do not vary with n, it follows that the nth term is a linear function of n. That is, the nth term is of the form

(4) $t_n = kn+m$, where k and m are constants.

Conversely, any function of the form (4) is the nth term of an arithmetic series which can be found as follows:

Since the nth term is given by (4) for any value of n, it follows that the $(n-1)$th term is

$$t_{n-1} = k(n-1)+m.$$

Therefore $$t_n - t_{n-1} = k, \text{ a constant.}$$

Hence the series is arithmetic, with the common difference k and with the first term

$$t_1 = k+m.$$

Formula (3) shows the dependence of l on n. It should be noted, however, that this formula involves four numbers, a, n, d, and l, any one of which may be considered as a function of the other three. If any three of these numbers are known, the fourth can be determined.

EXAMPLE 1. Find the 25th and rth terms of the arithmetic series 9, 7, 5, $\cdots$.

For the given series, we have $a = 9$ and $d = -2$. Therefore from formula (2),

$$t_{25} = 9+24(-2) = -39,$$

and $$t_r = 9+(r-1)(-2) = 11-2r.$$

Note that if $r = 25$, the value of t_r becomes $11-50$ or -39, which checks the answers.

EXAMPLE 2. If a, l, and d have the values 5, -52, and -3, respectively, find n.

By substituting in formula (3), we obtain

$$-52 = 5+(n-1)(-3).$$

Therefore $$n = 20.$$

Example 3. Given that the 6th and 30th terms of an arithmetic series are, respectively, 17 and 65, find the series.

From formula (2),

$$17 = a+5d$$

and

$$65 = a+29d.$$

Therefore, $a = 7$ and $d = 2$, and the series is

$$7,\ 9,\ 11,\ 13,\ \cdots.$$

EXERCISES

1. Determine which of the following are arithmetic series:

(i) $7,\ 12,\ 17,\ \cdots;$

(ii) $2,\ -1,\ -4,\ \cdots;$

(iii) $4,\ -6,\ 8,\ -10,\ \cdots;$

(iv) $x+y,\ 2x,\ 3x-y,\ \cdots;$

(v) $\sqrt{3},\ 1/\sqrt{3},\ -1/\sqrt{3}\ \cdots;$

(vi) $2m+n,\ 3m+3n,\ 4m+9n,\ \cdots.$

2. For each of the arithmetic series of ex. 1, find the values of a, d, and the next two terms.

3. Find the 40th and the kth terms of $5,\ 9,\ 13,\ \cdots.$

4. Find t_{10}, t_{100}, t_{1000}, and t_m for the series $3,\ 1,\ -1,\ \cdots.$

5. For the series $1,\ 5/3,\ 7/3,\ \cdots$, find a, d, t_{31}, and t_x.

6. Given the arithmetic series $0.8,\ -0.4,\ -1.6,\ \cdots$, find t_{16} and t_x. Show that t_x is a linear function of x.

7. If a, l, and d have the values 7, 65, and 2, respectively, find n.

8. Which term of the arithmetic progression 3, 9, 15, $\cdots$ is 117?

9. How many multiples of 6 are there between 100 and 200?

10. The 20th term of an arithmetic series is -113 and the 30th term is -173. Find a and d.

11. In an arithmetic series, the 41st and 15th terms are 30 and -9, respectively. Find t_{19} and t_{23}.

12. The 10th term of an arithmetic progression is -20 and the 17th term is -41. Find a and d, and write the first three terms of the series. Find also the pth term.

13. If the series $m+1$, $6m-5$, $8m+10$, $\cdots$ is arithmetic, find m.

14. Find t_n for the series k, $k-2$, $k-4$, $\cdots$. Use the expression so obtained to get the values of t_{39}, t_{50}, and t_{100}.

15. In an arithmetic series of 20 terms, $a=4$ and $l=99$. Write the first four and the last four terms of this series.

16. The common difference of an arithmetic series is 3.2 and the 85th term is 278.8. Find a.

17. Which term of the series 8, 13, 18, $\cdots$ is 503?

18. If the first term of an arithmetic series is $5b$ and the common difference, $2b$, find t_{101} and t_{2n}.

19. Plot the linear function $y=2x+3$ by giving to x the integral values 0 and 7. Show from the graph that the values of the function for consecutive integral values of the independent variable from 0 to 7 form an arithmetic series. Read from the graph the values of a and d for the series.

20. In formula (3), does doubling n double the nth term? Does doubling d double the nth term? Does doubling a and d
d change when the value of $l-a$ is doubled?
double the nth term? If n is a constant other than 1, how does

43. Arithmetic means. *If three numbers a, x, b are in arithmetic progression, x is called the* ***arithmetic mean*** *between a and b.* The value of x in terms of a and b is given by a simple expression. For, if a, x, b form an arithmetic series, then

$$x-a=b-x.$$

Therefore

$$2x=a+b,$$

and

$$x=\frac{a+b}{2}. \tag{5}$$

The arithmetic mean between two numbers is the average which is commonly used in arithmetic.

43.1. A related problem of a more general sort is that of inserting between a and b, k numbers which, together with a and b, will form an arithmetic series of $k+2$ terms. In this series, the terms between a and b are called the *arithmetic means.* The following example illustrates the method of determining such means.

EXAMPLE. Insert 5 arithmetic means between 21 and 3.

In the completed series we have

21, 5 inserted means, 3.

This forms an arithmetic series in which 21 is the first term and 3 the seventh term. Therefore, if the common difference is denoted by d,

$$3=21+6d$$

or

$$6d=-18$$

and

$$d=-3.$$

Thus, the arithmetic means are 18, 15, 12, 9, 6.

EXERCISES

1. Find the arithmetic mean between

   (*a*) 21 and -3, (*b*) $\sqrt{5}$ and $3\sqrt{5}$, (*c*) $x+a$ and $x-a$.

2. Insert 5 arithmetic means between 7 and 31.

3. Insert 4 arithmetic means between $1/\sqrt{2}$ and $3\sqrt{2}$.

4. Insert 7 arithmetic means between k and $9k-8a$.

5. Two fence posts are in line with a tree and on the same side of it, the first being 20 feet and the second 80 feet from the tree. It is desired to insert 4 posts between the two and in a straight line with them so that all the posts are equally spaced. At what distances from the tree should they be inserted?

44. **Sum of an arithmetic progression.** The following example illustrates a method of finding the sum of a given number of terms of an arithmetic series.

If $S=4+9+14+19+24+29,$

then $S=29+24+19+14+9+4.$

Hence $2S=33+33+33+33+33+33,$

or $2S=33\times 6.$

Thus $S=\frac{1}{2}(33\times 6)=99.$

We shall now use this method to find the sum of n terms of the general arithmetic series

Given the arithmetic series

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots \text{ to } n \text{ terms},$$

to find its sum,* s_n.

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*We shall use the notation s_1, s_2, s_3, $\cdots$, s_n to denote the sum of 1 term, 2 terms, 3 terms, $\cdots$, n terms, respectively.

If the nth term is denoted by l, we have

$$s_n=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l,$$
$$s_n=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a.$$

Hence

$$2s_n=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)+(a+l).$$

Combining the terms on the right, we obtain

$$2s_n=n(a+l),$$

or

$$\boxed{s_n=\frac{n}{2}(a+l).} \tag{6}$$

Since, from formula (3),

$$l=a+(n-1)d,$$

it follows that

$$\boxed{s_n=\frac{n}{2}\left\{2a+(n-1)d\right\}.} \tag{7}$$

Formula (6) is useful when the last term is known and formula (7) makes use of the common difference.

44.1. The sum of n terms of an arithmetic progression is seen to be a quadratic function of n, since from formula (7)

$$s_n=\frac{n}{2}\left\{2a+(n-1)d\right\}=\frac{1}{2}dn^2+\frac{1}{2}(2a-d)n.$$

If a and d are known constants, the sum of n terms is of the form

$$s_n=pn^2+qn, \tag{8}$$

where p and q are constants. Conversely, any function of this form is the sum of n terms of some arithmetic series. The method used in the following example can be applied to the general case.

EXAMPLE 1. Find the series for which $s_n = 6n^2+9n$, and prove it is arithmetic.

Because $s_n=6n^2+9n$,

therefore $s_{n-1}=6(n-1)^2+9(n-1)$.

Hence $s_n-s_{n-1}=t_n=12n+3$.

It follows that $t_{n-1}=12(n-1)+3$,
and therefore $t_n-t_{n-1} = 12$, a constant.

Thus the series is arithmetic, having $d=12$, and $a=s_1=15$.

44.2. Further uses of the sum of n terms of an arithmetic progression are illustrated in the following examples.

EXAMPLE 2. Find the sum of (*a*) $1+2+3+\cdots+100$;

(*b*) $2+4+6+\cdots$ to 100 terms.

(*a*) Since $a=1$, $l=100$, and $n=100$, formula (6) gives

$$s_{100}=50(1+100)=5050.$$

(*b*) We have $a=2$, $d=2$, and $n=100$. Therefore we obtain from formula (7)

$$s_{100}=50(4+99\times2)=50\times202=10{,}100.$$

EXAMPLE 3. How many multiples of 7 are there between 100 and 1000? What is the sum of these multiples?

The smallest multiple of 7 between 100 and 1000 is 105, and the largest is 994. Denote the number of these multiples by n. They give rise to an arithmetic series with first term 105, common difference 7, and nth term 994. Therefore, from (2),

$$994=105+(n-1)(7).$$

This gives $n=128$. The sum of the multiples is now found from (6). We have

$$s_{128}=64(105+994)=70{,}336.$$

EXAMPLE 4. How many terms of the arithmetic series 25, 20, 15, $\cdots$ must be taken to yield the sum 70?

If we let n denote the required number of terms, we have from formula (7)

$$70=\frac{n}{2}\{50+(n-1)(-5)\}.$$

This gives the equation

$$n^2-11n+28=0,$$

whence $$n=7 \text{ or } 4.$$

Both of these values for n are admissible, and the reason for the two solutions is made clear by writing out the first 7 terms of the series, thus:

$$25,\ 20,\ 15,\ 10,\ 5,\ 0,\ -5.$$

In problems of this type, only positive, integral values of n are admissible solutions.

EXAMPLE 5. Find the sum of the m arithmetic means between x and y.

The completed series will be an arithmetic progression whose first term is x and whose $(m+2)$th term is y. The sum of the $m+2$ terms of this series is obtained from formula (6) to be

$$s_{m+2}=\frac{m+2}{2}(x+y).$$

Therefore, the sum of the m means is

$$s_{m+2}-(x+y)=\frac{m}{2}(x+y).$$

EXERCISES ON THE ARITHMETIC PROGRESSION

A

1. Find the sum of (*a*) $4+7+10+\cdots$ to 20 terms,
(*b*) $6+4+2+\cdots$ to 50 terms.

2. Find the sum of 50 terms of an arithmetic series of which the first term is 2 and the 50th term is 149.

3. Find the sum of (*a*) $0.1+0.3+0.5+\cdots$ to p terms,
(*b*) $-2-7-12-\cdots$ to 15 terms.

4. For the series $7.5+5.0+2.5+\cdots$, find s_{40}, s_{100}, and s_x. Show that s_x is a quadratic function of x.

5. Insert 7 arithmetic means between 3 and 15 and find their sum.

6. If 100 arithmetic means are inserted between 1 and 1000, what is their sum?

7. How many odd integers lie between 50 and 170? Find their sum.

8. What is the value of $8+12+16+\cdots+60$?

9. What is the sum of all the multiples of 5 between 100 and 200?

10. If the sum of 8 terms of an arithmetic series is 52 and the first term is 24, find the common difference. What is the 6th term?

11. Find the sum of $6r+(7r+s)+(8r+2s)+\cdots$ to $2n$ terms.

12. How many terms of the series $4, 0, -4, \cdots$ must be taken to yield the sum $-19{,}796$?

13. How many terms of the series $56, 49, 42, \cdots$ must be taken to yield the sum 245? Explain the two answers.

14. Find the sum of 12 terms of the series 3, $3\frac{1}{2}$, 4, $\cdots$ (*a*) by first finding the 12th term and then using formula (6), (*b*) without using the value for the 12th term.

15. If the bell on the town clock strikes the hours, how many times a day does the clapper strike the bell?

B

16. If the first term of an arithmetic series is $6c$ and the common difference, $2d$, find s_r.

17. Find the sum of n terms of 10, 5, 0, $\cdots$. Use the function of n so obtained to get the sum of 15 terms of this series.

18. The sum of 10 terms of an arithmetic series is 500, and the sum of 24 terms is 1872. Find a and d.

19. The sum of 8 terms of an arithmetic progression is 52 and the sum of 16 terms is -88. Find the 4th term.

20. In winding a string around a cone, it is found that the first 3 turns require 1 inch, $1\frac{3}{4}$ inches, and $2\frac{1}{2}$ inches, respectively. How many feet of string are needed for 65 turns?

21. When a body falls from rest, it drops about 16 feet in the 1st second, 48 feet in the 2nd second, 80 feet in the 3rd second, and so on. If a stone were dropped from a balloon floating at a height of 10,000 feet, how long would it take the stone to reach the ground? (Neglect air resistance.)

22. The sum of the first 4 terms of an arithmetic progression is 19, and the sum of the next 4 terms is 43. Find the sum of the first 12 terms.

23. If the sum of n terms of an arithmetic series is 1000, and the arithmetic mean between the first and last terms is 100, find the number of terms.

24. Along a straight line 10 potatoes are placed at equal intervals, the length of each interval being 6 feet. A basket is placed at the end of the line 30 feet from the first potato. A boy starting from the basket, picks up the potatoes and carries them back, one at a time, to the basket. What is the total distance covered by the boy in the potato race?

25. If the sum of x terms of a series is $3x^2+2x$, show that the series is arithmetic. Find the sum of 10 terms (*a*) by using formula (7), (*b*) by using the given function of x directly.

26. For the series $22, 16, 10, \cdots$, find (*a*) the number of the first negative term, and (*b*) the least number of terms which must be taken to yield a negative sum.

27. Without using formulas (6) or (7), find the sum of 40 terms of the series $18+14+10+6+\cdots$.

28. Show how to insert m arithmetic means between p and q, and give the formula for the rth of these means.

29. If the numbers p, q, r are in arithmetic progression, prove that (*a*) $p+k, q+k, r+k$ and (*b*) pk, qk, rk are in arithmetic progression, where k is any number; also, if $k\neq 0$, show that the series $p/k, q/k, r/k$ is arithmetic.

30. In order to stop a locomotive, the power is shut off and the brakes applied. In the first second after this is done, the drive-wheels make 11 revolutions; in the second second, they make $10\frac{2}{3}$ revolutions; in the third second, $10\frac{1}{3}$ revolutions, and so on. If the locomotive comes to rest at the end of the second in which the drive-wheels make 1/3 revolution, how long does it take to stop? If the circumference of the drive-wheel is 8 feet, how far forward does the locomotive go while coming to rest?

31. If the middle term of an arithmetic progression of 11 terms is 20, find its sum.

32. Seven circles are drawn with radii 1, 3, 5, 7, 9, 11, and 13 units, respectively. Do the lengths of the circumferences of these circles, taken in the same order, form an arithmetic series? Do the measures of their areas?

33. Show that the series formed by taking every pth term of a given arithmetic series, is also arithmetic.

34. If the rth term of a series is $13-3r$, show that the series is arithmetic. What is the number of the first negative term in the series? For what values of n is s_n positive?

35. Find a formula for d in terms of l, a, and n; also, find a formula for d in terms of s_n, n, and a.

45. The harmonic progression. *If each term in an arithmetic progression is replaced by its reciprocal, the resulting series is called a harmonic progression.* For example, the numbers 3, 5, 7, 9 form an arithmetic series; hence, the numbers 1/3, 1/5, 1/7, 1/9 form a harmonic series. The arithmetic series from which the harmonic series is derived is called the *corresponding arithmetic series.*

45.1. Harmonic means. If three numbers a, x, b form a harmonic progression, x is called the harmonic mean between a and b. The value of x is readily found in terms of a and b since, if x is the harmonic mean between a and b, it follows that $1/x$ is the arithmetic mean between $1/a$ and $1/b$. Therefore, we have

$$\frac{1}{x}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right),$$

or

$$\frac{1}{x}=\frac{a+b}{2ab}.$$

Hence

(9) $$x=\frac{2ab}{a+b}.$$

As in the case of the arithmetic series, a more general problem of a similar type is that of inserting k numbers between a and b so that the k numbers, together with a and b, will form a harmonic series of $k+2$ terms. The k numbers between a and b are called *harmonic means.*

45.2. Problems involving a harmonic progression are generally solved by using the corresponding arithmetic progression. It should be noted, however, that *there is no simple formula for the sum of n terms of a harmonic progression;* this sum is *not* obtained by finding the sum of n terms of the corresponding arithmetic series and taking its reciprocal.

EXAMPLE 1. Show that 2, 3, 6 are the first three terms of a harmonic progression and find its 16th term.

Since, by formula (9), the harmonic mean between 2 and 6 is

$$\frac{2\times 2\times 6}{2+6}=3,$$

it follows that 2, 3, 6 form a harmonic series. This fact may also be shown by noting that the reciprocals of these numbers, namely 1/2, 1/3, 1/6, form an arithmetic series whose first term is 1/2 and whose common difference is $-1/6$. The 16th term in this corresponding arithmetic series is given by (2) to be

$$\frac{1}{2}+15\left(-\frac{1}{6}\right)=-2.$$

The 16th term in the harmonic series is, therefore, $-1/2$.

EXAMPLE 2. Insert 5 harmonic means between 420 and 60.

In the corresponding arithmetic series, we insert 5 arithmetic means between 1/420 and 1/60. Since 1/60 is the 7th term in the completed arithmetic series, from (2) we have

$$\frac{1}{60}=\frac{1}{420}+6d, \text{ whence } d=\frac{1}{420}.$$

Hence, the arithmetic means are

$$\frac{1}{210}, \frac{1}{140}, \frac{1}{105}, \frac{1}{84}, \frac{1}{70};$$

and the required harmonic means are

$$210, \quad 140, \quad 105, \quad 84, \quad 70.$$

EXERCISES ON THE HARMONIC PROGRESSION

A

1. Prove that the following are harmonic progressions and find the next two terms of each:

(*a*) 6, 3, 2;

(*b*) 150, 100, 75, 60;

(*c*) −3, −3/2, −1.

2. Find the harmonic mean of 5 and 10; of $a+b$ and $a-b$.

3. Insert (*a*) 3 harmonic means between −120 and −24, (*b*) 8 harmonic means between 10/11 and 1/2.

B

4. The 4th term of a harmonic progression is 1/2, and the 10th term is −1/2; find the first term and the sum of the first 5 terms.

5. Find the 50th term in the harmonic series 3, 4, 6, ... Has this series a 5th term?

6. In a harmonic series, the first term is twice the second and the third term is 40. Find the first 6 terms.

7. The first term of a harmonic progression is 2 and the sum of 3 terms is 11/3. Find the series.

8. The harmonic mean of two numbers is 5 and the arithmetic mean is 9. Find the numbers.

9. Two towns are 10 miles apart. An automobile travels from one town to the other at an average speed of 60 miles per hour, and returns at an average speed of 20 miles per hour. Show that the average speed in miles per hour for the whole trip is the harmonic mean of 60 and 20.

10. If the harmonic mean of two numbers is 16/25 of their arithmetic mean, find the ratio of the numbers.

46. The geometric progression. *A geometric series or geometric progression* is a series in which each term, after the first, is obtained by multiplying the preceding term by a fixed number, called the* ***common ratio.*** It follows that the common ratio of a geometric progression is found by dividing any term into the one immediately following it in the series. For example, 2, 4, 8, 16 is a geometric series with common ratio 2; also $a, -ab, ab^2, -ab^3$ is a geometric series with common ratio $-b$. It is evident that any geometric series can be represented by

$$a, ar, ar^2, ar^3, \cdots, \tag{10}$$

in which the first term is denoted by a and the common ratio by r. We shall rule out the trivial cases in which $a=0$ or $r=0$ or $r=1$.

46.1. General term of a geometric progression. An inspection of the general series (10) reveals the fact that the index of r is always *one less than the number of the term.* For example,

$$t_{50}=ar^{49}.$$

Similarly

$$\boxed{t_n = l = ar^{n-1}.} \tag{11}$$

---

*Sometimes the abbreviation G.P. is used for "geometric progression". Likewise H.P. is used for "harmonic progression".

If, in a given geometric series, a and r are known constants, formula (11) gives the nth term as a function of n. It should be noted that ar^n is also the nth term of a geometric progression whose first term is not a, but ar. Conversely, it can be shown that any function of the form kp^n, where k and p are constants, is the nth term of a geometric series. For, we have

$$t_{n-1}=kp^{n-1}.$$

Then, by dividing t_n by t_{n-1}, we obtain the quotient p, a constant. Therefore, the series is geometric with its first term kp, and common ratio p.

46.2. The contrast between the arithmetic and the geometric progression is now apparent: in the former, the terms grow or diminish by addition; in the latter, by multiplication. The arithmetic series of which the first two terms are 1, 2, is 1, 2, 3, 4, 5, $\cdots$, while the geometric series with the same first two terms is 1, 2, 4, 8, 16, $\cdots$. The surprising difference in the rates of increase of the terms is shown by the fact that the 10th term of the arithmetic series is 10, while the 10th term of the geometric series is 512. A practical example of the arithmetic law of growth is afforded by the successive yearly amounts of a sum of money invested at simple interest. For example, \$100 bearing simple interest at 5% per annum amounts to \$105 in one year, \$110 in two years, \$115 in three years, and so on. However, as is explained in chapter X, if the 5% per annum were compounded annually, the successive amounts would grow by geometric progression and would be $\$100(1.05)$, $\$100(1.05)^2$, $\$100(1.05)^3$ at the end of one, two, and three years, respectively.

EXAMPLE. If the 5th term of a geometric series is 324 and the 7th term is 2916, find the nth term. Which term is 108?

Applying formula (11), we obtain

$$324=ar^4,$$

$$2916=ar^6.$$

By division, we have $r^2 = 9$, whence $r = 3$ or -3. In either case we find by substitution that $a = 4$. Hence,

$$t_n = 4 \cdot 3^{n-1}, \text{ if } r = 3,$$

and

$$t_n = 4\,(-3)^{n-1}, \text{ if } r = -3.$$

If we denote the number of the term 108 by n, we have

$$108 = 4 \cdot 3^{n-1}, \text{ if } r = 3,$$

and

$$108 = 4\,(-3)^{n-1}, \text{ if } r = -3.$$

In the first case, $3^{n-1} = 27$, so that $n-1 = 3$ and $n = 4$. However, for the series in which $r = -3$, there is no term 108, because no positive, integral value of n will make $(-3)^{n-1}$ yield 27. The two series to which the problem gives rise are

$$4, \quad 12, \quad 36, \quad 108, \quad 324, \cdots,$$

and

$$4, \quad -12, \quad 36, \quad -108, \quad 324, \cdots.$$

EXERCISES

1. Determine which of the following series are geometric:

(*a*) $1, 3, 9, \cdots;$

(*b*) $80, -40, 20, -10, \cdots;$

(*c*) $4, 8, 12, \cdots;$

(*d*) $2/3, 1/2, 3/8, \cdots;$

(*e*) $2r, 6r^2, 18r^3, 54r^4, \cdots;$

(*f*) $p-q, p^2-q^2, p^3-q, \cdots.$

2. Write down the next two terms of each geometric series in ex. 1.

3. Find the 9th term of the geometric progression 729, $-243, 81, \cdots$.

4. Write the 100th and the kth terms of the series 1, 5, 25, $\cdots$.

5. Find the 10th and the pth terms of 64, -32, 16, $\cdots$.

6. Find t_{20}, t_{100}, and t_n for the series $m^3, m^5, m^7, \cdots$.

7. Which term of the series 729, 243, 81, $\cdots$ is $1/27$?

8. If the 3rd term of a geometric progression is 4 and the 10th term is $32\sqrt{2}$, find the first 2 terms of the series.

9. Find the common ratio of a geometric series whose 4th term is 135 and whose 7th term is 3645.

10. The first term of a geometric series is 972 and the 5th term is 12. Find the series.

11. If the 5th and 8th terms of a geometric progression are 256 and 16,384, respectively, find a and r.

12. Find the 5th and 10th terms of the series $x, 2x^2, 4x^3, \cdots$. Which term is $1024\, x^{11}$?

13. Find (*a*) the 8th term of the series 5, 5/2, 5/4, $\cdots$;
(*b*) the 12th term of the series 3, 0.3, 0.03, $\cdots$.

14. Which term of the geometric progression 4096, 1024, 256, $\cdots$ equals 1?

15. The common ratio of a geometric series is 1/6 and its 100th term is 3000. What are the 99th and 101th terms of this series?

16. The kth term of a series is $3 \cdot 5^k$. Prove that the series is geometric; find a and r.

17. Find the pth term of the series $m+5$, $2m^2+10m$, $4m^3+20m^2, \cdots$.

18. It is said that the number of bacteria in milk doubles every 3 hours. By how much does the number of bacteria increase in (*a*) 12 hours? (*b*) $3n$ hours?

19. If, in formula (11), a and r are constants and $0<r<l$, how does an increase in n affect l? If r and n are constants, what is the effect on l if a is doubled? If a and l are positive constants, how does an increase in n affect r (*a*) if $l>a$, (*b*) if $l<a$?

47. **Geometric means.** *If a, x, b form a geometric series, x is called a geometric mean between a and b.* As in the cases of the arithmetic and harmonic series, the idea of a geometric mean is extended to include k numbers inserted between a and b so that a, the k inserted numbers, and b form a geometric series of $k+2$ terms.

A simple expression for x in terms of a and b is found from the fact that, since a, x, b are in geometric progression,

$$\frac{x}{a}=\frac{b}{x}, \text{ or } x^2=ab.$$

Therefore

$$x=\pm\sqrt{ab}. \tag{12}$$

Formula (12) gives two geometric means between a and b. However, if a and b are opposite in sign, the product under the root sign is negative, and there is no real geometric mean. When a and b have the same signs, the geometric mean which is the useful average is the one whose sign is the same as that of a and b. Therefore, if a and b are both positive, we shall define *the* geometric mean between them to be $\sqrt{ab}$; and, if a and b are both negative, we shall define *the* geometric mean between them to be $-\sqrt{ab}$. A geometric mean between a and b is also a mean proportional between them.

EXAMPLE 1. The geometric mean between 2 and 8 is $\sqrt{16}=+4$.

Example 2. The geometric mean between -2 and -8 is $-\sqrt{16}=-4$.

Example 3. Insert 3 geometric means between 3 and 243.

The completed geometric series consists of

3, 3 inserted geometric means, 243.

Hence, in the completed series, we have

$$a=3, \text{ and } t_5=ar^4=243.$$

Therefore, $$r^4=81 \text{ or } r=\pm 3.$$

Thus, the means are ± 9, 27, ± 81, the upper signs corresponding to $r=3$, and the lower signs to $r=-3$.

★*47.1.* Definitions have been given for the arithmetic and geometric means of *two* numbers. These definitions are now extended to the general case of n numbers as follows: *The arithmetic mean of n numbers is their sum divided by n,* and *the geometric mean of n positive numbers is the positive nth root of their product.* The average which is commonly used in arithmetic is the arithmetic mean. However, there are cases in which the geometric mean is the significant average. Again, when a number is desired which will be representative of a statistical set of numbers, the geometric mean has a property which gives it, in some cases, an advantage over the arithmetic mean. The following examples illustrate these points.

Consider first the three numbers 1, 2, 4. Their arithmetic mean is $\frac{1+2+4}{3}=2\frac{1}{3}$, and their geometric mean $\sqrt[3]{1\times 2\times 4}=2$. Now replace the number 1 by 27, so that the three numbers are 27, 2, 4. Their arithmetic mean is $\frac{27+2+4}{3}=11$, and their geometric mean $\sqrt[3]{27\times 2\times 4}=6$. The presence of a single number differing considerably from the others has a more marked effect on the arithmetic mean than on the geometric. In a problem in which the numbers represent observations, it may be suspected that a few numbers which deviate greatly from the mean are affected by experimental errors. In such a case, the geometric mean may be more representative of the numbers of the group inasmuch as it is less affected by extreme deviations of one or more numbers.

As a second example, consider the following problem in the fluctuation of prices. Suppose that a certain article sold for \$1 in 1928, and that in each succeeding year the change in price is recorded as a percentage of the previous year's price as follows: 1929, 20% increase; 1930, 10% decrease; 1931, 32.5% decrease. The numbers which give the ratios of the prices in successive years are 1.2, 0.9, and 0.675. The arithmetic mean of these numbers is 0.925, which indicates an average decrease of 0.075 or $7\frac{1}{2}\%$. On the other hand, the geometric mean is $\sqrt[3]{1.2\times0.9\times0.675}=\sqrt[3]{6/5\times9/10\times27/40}=0.9$, which indicates an average decrease of 10%. Which of these is the significant average in this problem? Since the 1931 price is $1.2\times0.9\times0.675$ of \$1 which is the same as $0.9\times0.9\times0.9$ of \$1, it is clear that the geometric mean fits the problem and the arithmetic mean does not.

EXERCISES

1. Find the geometric mean between (*a*) 3 and 27, (*b*) $-1/2$ and $-1/8$, (*c*) $\sqrt{2}-1$ and $\sqrt{2}+1$, (*d*) $-3/2$ and $3/8$.

2. Insert (*a*) 5 geometric means between 81/64 and 1/9; (*b*) 7 geometric means between 3 and 768.

3. If $p-4$, $2p+4$, $9p+6$ is a geometric series, find p.

4. Insert m geometric means between h and k and obtain a formula for the pth of these means.

5. What is the geometric mean of the numbers

$$\tfrac{1}{3},\ 1,\ 3,\ 3,\ 9,\ 27?$$

6. Prove that if a square is equal in area to a given rectangle, the side of the square, measured in any units, is equal to the geometric mean of the length and breadth of the rectangle, measured in the same units.

★7. What average rate of increase is equivalent to two successive price increases of 20% and 10%?

★8. What average rate of increase is equivalent to an increase of 25% followed by a decrease of 20%?

★9. A merchant is allowed trade discounts of 10% and 37.5% on goods purchased. What average rate of discount is equivalent?

48. Sum of a geometric progression. Given the series $a, ar, ar^2, ar^3, \cdots$, we wish to find a formula for the sum of n terms, s_n. The required expression is obtained by multiplying each term of the series by r and subtracting the new series so formed from the given series. Thus,

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}.$$

Hence $$rs_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n,$$

and $$(1-r)s_n = a \qquad\qquad\qquad\qquad - ar^n.$$

Therefore,

(13*a*) $$\boxed{s_n = \frac{a(1-r^n)}{1-r}.}$$

If we multiply numerator and denominator of (13*a*) by -1, we have the equivalent formula

(13*b*) $$\boxed{s_n = \frac{a(r^n-1)}{r-1}.}$$

If a and r are known constants, this formula gives the sum of a geometric series as a function of n. Formula (13a) is more convenient when r is numerically less than 1, that is, if $|r|<1$. If $r=1$, the formula does not apply. However, in this case, the series becomes $a, a, a, a, \cdots$ and the sum of n terms is seen to be na.

48.1. The following examples illustrate the use of formula (13).

Example 1. Find the sum of $256-128+64-32+\cdots$ to 12 terms. How many terms of this series must be taken to give the sum 171?

The series is geometric with $a=256$ and $r=-1/2$. Therefore, substituting in (13a), we have

$$s_{12}=\frac{256\{1-(-\frac{1}{2})^{12}\}}{1-(-\frac{1}{2})}=\frac{512}{3}\{1-(\tfrac{1}{2})^{12}\}$$

$$=\frac{512}{3}-\frac{1}{24}=170\tfrac{5}{8}.$$

If we let n denote the number of terms required to give the sum 171, formula (13a) gives

$$171=\frac{256\{1-(-\frac{1}{2})^{n}\}}{1-(-\frac{1}{2})}=\frac{512}{3}\{1-(-\tfrac{1}{2})^{n}\}.$$

Thus

$$(-\tfrac{1}{2})^{n}=1-\frac{513}{512}=-\frac{1}{512}=(-\tfrac{1}{2})^{9}.$$

Therefore $n=9$.

Example 2. Prove that the series for which $s_r=5(2^r-1)$ is geometric.

Since $s_r=5(2^r-1)$,

it follows that $s_{r-1}=5(2^{r-1}-1)$.

Hence

$$t_r = s_r - s_{r-1} = 5\,(2^r - 2^{r-1}) = 5 \cdot 2^{r-1}.$$

Therefore $$t_{r-1} = 5 \cdot 2^{r-2}$$

The quotient of t_r divided by t_{r-1} is 2, a constant. Therefore, the series is geometric with common ratio 2. The first term $= s_1 = 5$.

EXAMPLE 3. Use the formula for the sum of n terms of a geometric progression to prove that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}).$$

The second factor of the right member is the sum of a geometric series of n terms with common ratio b/a. By (13), this sum is equal to

$$\frac{a^{n-1}\left\{1-\left(\frac{b}{a}\right)^n\right\}}{1-\left(\frac{b}{a}\right)} = \frac{a^n\left\{1-\left(\frac{b}{a}\right)^n\right\}}{a-b}$$

$$= \frac{a^n - b^n}{a-b}.$$

Hence, when this sum is multiplied by $a-b$, the product is $a^n - b^n$, which was to be proved.

EXERCISES

1. Find the sum of the following geometric series:

   (a) $4+20+100+\cdots$ to 6 terms,

   (b) $3-6+12-\cdots$ to 10 terms.

2. Find the sum of (a) $1+3+9+\cdots$ to 12 terms,

   (b) $1+1/2+1/4+\cdots$ to 9 terms.

3. Find the sum of $1-x+x^2-x^3+\cdots$ to 20 terms.

4. Find the sum of $1+2p+4p^2+\cdots+64p^6$.

5. What is the value of $5-10+20-40+\cdots+5120$?

6. If the sums of the first 9 terms of the geometric progressions

$$16+\frac{16}{5}+\frac{16}{25}+\cdots \text{ and } 25-\frac{25}{4}+\frac{25}{16}-\cdots$$

are denoted by p and q respectively, find which of p and q is the greater.

7. The sum of the first 4 terms of a geometric series is 4 and the sum of the first 8 terms is 328. Find r and the first 3 terms of the series.

8. An exhaust pump removes 1/3 of the air from a closed tank at each stroke. What fraction of the original amount of air is left in the tank after 8 strokes?

9. Without using a formula, find the sum of 8 terms of the series $1-2/3+4/9-\cdots$.

10. Find the sum of $2k$ terms of a series of which the first term is 1, every even-numbered term is formed by multiplying the term immediately preceding it by x, and every odd-numbered term, after the first, is formed by multiplying the term immediately preceding it by y.

11. A ball is dropped from a height of 16 feet. If, at each bounce, the ball rises to $\frac{3}{4}$ of the height from which it fell, how far has it travelled when it strikes the ground for the 6th time?

12. In a spirit of generosity, a father agrees to make a contribution to his son's penny-bank each week. He puts a 1 cent piece in the bank the first week, but tells his son that each week he will double the amount that he gave the week before. How much money would it take to keep up this scheme for 20 weeks?

13. A pendulum is brought to rest slowly by air resistance. If the tip of the pendulum sweeps out an arc of 100 centimetres on its first swing, and if each subsequent arc swept out is less than the preceding arc by one-tenth of it, what is the distance travelled by the tip of the pendulum in 5 swings?

14. What is the value of $\sqrt{3}+3+3\sqrt{3}+\cdots+81$?

49. The infinite geometric series. DEFINITION. *If the values of a variable u eventually become and remain numerically greater than any arbitrarily large number K, chosen in advance, then we say that* ***u increases without limit*** *or that* ***u becomes infinite.*** Consider n terms of the series

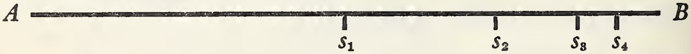
$$1,\ 2,\ 4,\ 8,\ 16,\ \cdots,$$

and allow n to increase without limit. It is clear that the terms become infinite, and, therefore, that the sum of n terms also becomes infinite. That is, s_n becomes infinite as n becomes infinite.

Next, consider the series of terms

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \cdots\ (\tfrac{1}{2})^{n-1}.$$

A useful representation of the sum of this series for different values of n may be obtained by the following graphical method.



Let AB be a line two units in length. Starting from the point A on this line, mark off successive distances measured by the terms of the series. Thus s_1, the mid-point of AB, is one unit from A so that the distance As_1 will represent the sum of one term. The second point s_2 is the mid-point of s_1B so that s_1s_2 equals one half unit and represents the second term. Therefore, As_2 is one and a half units and will represent the sum of two terms. Similarly, As_3 represents the sum of three terms, and so on. The successive points $s_1, s_2, s_3, \cdots$ approach B in such a way that as n becomes infinite, s_n comes arbitrarily close* to B. This fact is expressed by saying that the point s_n approaches the point B as its limit as n becomes infinite, or that the sum s_n (represented by As_n) approaches the limit 2 as n becomes infinite.

The fact needs emphasis that the sum of n terms of this series never reaches 2 for any value of n: the number 2 is not the sum of any number of terms, however great. It is *the limit towards which the sum of n terms tends as n becomes infinite.* The graphical representation above gives an intuitive picture of the situation. Inasmuch as each of the points $s_1, s_2, s_3, \cdots$ is midway between the preceding point and B, therefore no one of the points coincides with B.

Definition. *The symbol* $a_1 + a_2 + a_3 + \cdots$, *in which it is understood that each term has a term immediately following it, is called an infinite series.* Thus, in an infinite series, there is no last term.

The fact proved above is now expressed by saying that *the value of the infinite series*

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots$$

is 2, or

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots = 2.$$

The words *sum of an infinite series* are also used instead of *value of an infinite series,* but it is to be understood that this is an extension of the meaning of the word sum. In its original sense, the word sum applies only to a definite number of terms. In its extended sense, it applies also to the limit of the sum of n terms.

Definition. *Whenever, as in the example just considered, the sum of n terms of a series approaches a limit as n becomes infinite, the infinite series is said to converge or to be convergent; otherwise the series diverges or is divergent.*

---

*So close that, no matter how small a positive number k be chosen in advance, the distance Bs_n becomes and remains less than k.

49.1. It will now be shown that any infinite geometric series is convergent if the common ratio is numerically less than unity. In the series

$$a+ar+ar^2+\cdots,$$

take r such that $-1<r<1$. Then

$$s_n=a+ar+ar^2+\cdots+ar^{n-1}=\frac{a(1-r^n)}{1-r},$$

which may be written

$$s_n=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

We have to examine the change in this function of n as n increases without limit. Since the first term does not depend on n, it therefore suffers no change as n increases. In the second term, r^n is the only factor which depends on n. Now r is numerically less than 1, hence the value of r^n approaches zero as n becomes infinite.* It follows that the term

$$\frac{ar^n}{1-r}$$

approaches the limit zero as n becomes infinite, and therefore the limit of s_n is $\frac{a}{1-r}$. This is expressed by saying that the sum (in the extended sense explained above) of the infinite series is $\frac{a}{1-r}$. This fact we shall indicate by the notation

$$\boxed{s=\frac{a}{1-r},\quad(-1<r<1).}\tag{14}$$

Example 1. Find the sum of the infinite geometric series $25+5+1+\cdots$.

Since the common ratio is $1/5$, the series converges and its sum is given by (14).

---

*This will be evident to the student if he considers some examples. Thus: $(\frac{3}{4})^2=\frac{9}{16}=.56\cdots$; $(\frac{3}{4})^3=\frac{27}{64}=.42\cdots$; $(\frac{3}{4})^4=\frac{81}{256}=.32\cdots$; etc.

Therefore,

$$s = \frac{25}{1-\frac{1}{5}} = 31\tfrac{1}{4}.$$

Example 2. Find, if possible, the sum of the series

$$1 + \frac{4}{3} + \frac{16}{9} + \cdots .$$

Since r equals $4/3$, the terms increase without limit and the infinite series has no sum.

Example 3. A repeating decimal can be put into fractional form by expressing it as a convergent geometric series. Thus,

$$.\dot{6} = .666 \cdots = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \cdots$$

$$= \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{2}{3}.$$

EXERCISES

1. Determine which of the following infinite geometric series converge, and find the sums of those which are convergent.

(*a*) $6+4+8/3+ \cdots;$

(*b*) $.2-.02+.002- \cdots;$

(*c*) $1/3+1/2+3/4+ \cdots;$

(*d*) $\sqrt{2}+1+\frac{1}{\sqrt{2}}+ \cdots;$

(*e*) $5-5+5- \cdots .$

2. Find, as common fractions, the values of the following repeating decimals: (*a*) $.454545 \cdots$, (*b*) $2.377 \cdots$, (*c*) $.12080808 \cdots$.

3. The first term of an infinite geometric series is 16 and its sum is 64/7. Find r and write the first 3 terms of the series.

4. For what values of x does the series $1+3x+9x^2+\cdots$ have a sum?

5. In problem 13 on page 188, find the total distance travelled by the tip of the pendulum in coming to rest.

6. In problem 11 on page 187, find the total distance travelled by the ball in coming to rest.

EXERCISES ON THE GEOMETRIC PROGRESSION

A

1. Find the sum of the infinite series $0.7+0.07+0.007+\cdots$.

2. The common ratio of a geometric series is 5 and one of its terms is 1. Write the terms of the series which lie between 1000 and 0.001.

3. For the series $100(1.03)^4$, $100(1.03)^5$, $100(1.03)^6$, $\cdots$ write t_{20} and t_n.

4. For the series $30-90+270+\cdots$, find s_6 and s_∞.

5. The second term of a geometric series is 28 and the fourth term is 1372; find the fifth term.

6. The sum of 2 terms of a geometric series is 100 and the sum of 4 terms is 1000. Find a and r.

7. Insert 6 geometric means between 320 and 5/2 and find their sum.

8. Express as common fractions the following repeating decimals: (*a*) $0.444\cdots$; (*b*) $0.1515\cdots$.

9. How many terms of the series $1-3+9-\cdots$ must be added to yield -182?

10. Find, as a decimal fraction, the difference between 2 and the sum of 7 terms of the series $1+1/2+1/4+\cdots$.

11. What number must be added to each of the numbers 2, 16, 58, so that the resulting numbers form a geometric progression?

12. A rectangle is m feet long and n feet wide. Find the side of a square of equal area. Make a general statement of this result.

13. The first term of a geometric series is 13 and the 7th term is 832. Find r.

B

14. Find a short expression for $p^{10}+p^9q+\cdots+q^{10}$.

15. Find a geometric series such that the sum of its first and third terms is 52, and the sum of its second and fourth terms is 260.

16. It is related that the inventor of chess was offered any reward of his choosing by a grateful king. The inventor modestly asked for enough wheat so that he might have 1 grain for the first square of his chess-board, 2 grains for the second square, 4 grains for the third square and so on. How many grains of wheat are needed to complete this process? Use logarithms to find the number of digits in your answer.

17. The common ratio of a geometric progression is 1/2 and one of its terms is 1. Write down all the terms of this series which lie between 100 and .01 and find their sum.

18. The sum of the first 3 terms in a geometric series is 42, and the product, 512. Find a and r.

19. A chain letter is sent to 6 people, each of whom is requested to make 6 copies and send them to 6 friends, and so on. If one-half of those receiving letters of a set do as requested, how many letters have been written after 8 sets have been sent?

20. The first term of a geometric progression is 2, the last term 6250, and the sum 7812. Find the common ratio.

21. Prove that the series for which $s_p=3^p-1$ is geometric, and find the fifth term.

22. Use the formula for the sum of n terms of a geometric progression to prove that, if n is an odd, positive integer,

$$a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2- \cdots +b^{n-1}).$$

23. In an infinite geometric series, each term equals one-half the sum of all the terms that follow it, and the sum of the first three terms is 19. Find a and r.

24. The sum of an infinite geometric series is k times its first term. Find r.

25. The common ratio of a geometric progression is 2, the last term 896, and the sum 1785. Find the first term.

26. If the pth term of a series is $3 \cdot 5^p$, what is the kth term? What is the $(k-1)$th term? Is the series geometric? If so, find the values of a and r.

27. Does the series $6+8+32/3+128/9+ \cdots$ converge? If the 1001th term of this series were expressed in decimal notation, how many digits would there be in the integral part of the number?

28. Show that the reciprocals of the terms of a geometric series form another geometric series.

50. Series of powers of the natural numbers. The positive integers 1, 2, 3, $\cdots$ are also called the ***natural numbers.*** Since these numbers form an arithmetic progression, their sum to n terms is given by formula (6) as follows:

$$\boxed{1+2+3+\cdots+n=\frac{n(n+1)}{2}.} \tag{15}$$

A convenient notation for the sum of this series or of any series whose nth term is given as a function of n involves the use of Σ, which is the Greek letter S, the initial letter of the word sum. When the letter Σ appears before the nth term of a series, the whole symbol stands for the sum of n terms of the series;* for Σ we may read "the sum of all such terms as". Thus,

$$\Sigma n=1+2+3+\cdots+n,$$

$$\Sigma n^2=1^2+2^2+3^2+\cdots+n^2,$$

$$\Sigma n(n+1)=1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1).$$

50.1. A method of finding Σn without the use of the arithmetic series formula will now be given, and will then be generalized so as to find other sums. We use the identity

$$x^2-(x-1)^2=2x-1.$$

This equation, being an identity, is true for all values of x. By giving to x in succession the values 1, 2, 3, . . . , n, we get the following set of equations:

---

*Instead of $\sum f(n)$, a more flexible notation is $\sum\limits_{r=1}^{n} f(r)$ or $\sum\limits_{1}^{n} f(r)$, which means the sum of all terms obtained from $f(r)$ by giving to r the values 1, 2, 3, $\cdots$, n. An example will show the advantage of the latter notation. Thus $\sum\limits_{1}^{9} r=1+2+3+\cdots+9$ and $\sum\limits_{1}^{3} r^2=1+4+9$, whereas $\sum 9$ can be given no clear meaning. However, the notation $\sum f(n)$ will be sufficient for our purposes.

$$\begin{array}{rcll}
1^2- & 0^2 & =2\cdot 1 & -\ 1\,, \\
2^2- & 1^2 & =2\cdot 2 & -\ 1\,, \\
3^2- & 2^2 & =2\cdot 3 & -\ 1\,, \\
\multicolumn{4}{c}{\dotfill} \\
(n-1)^2- & (n-2)^2 & =2(n-1) & -\ 1\,, \\
n^2- & (n-1)^2 & =2\cdot n & -\ 1\,. \\
\hline
\end{array}$$

Hence, by addition,

$$n^2= 2(1+2+3+\cdots+n)-n,$$

or

$$n^2= 2\Sigma n-n,$$

which gives

$$\Sigma n=\frac{n^2+n}{2}=\frac{n(n+1)}{2}\,.$$

This is formula (15) above.

The justification of the use of this method is that it may be generalized to find the sums Σn^2, Σn^3, etc., which are not arithmetic series. Thus, to find Σn^2, we begin with the identity

$$x^3-(x-1)^3=3x^2-3x+1.$$

Giving to x the values 1, 2, 3, . . . , n, we get

$$\begin{array}{rcllr}
1^3- & 0^3 & = 3.1^2 & -\ 3.1 & +\ 1\,, \\
2^3- & 1^3 & = 3.2^2 & -\ 3.2 & +\ 1\,, \\
3^3- & 2^3 & = 3.3^2 & -\ 3.3 & +\ 1\,, \\
\multicolumn{5}{c}{\dotfill} \\
(n-1)^3- & (n-2)^3 & = 3(n-1)^2 & -3(n-1) & +\ 1\,, \\
n^3- & (n-1)^3 & = 3.n^2 & -\ 3.n & +\ 1\,. \\
\hline
\end{array}$$

By addition, $$n^3=3\Sigma n^2-3\Sigma n+n.$$

Hence
$$\begin{aligned}3\Sigma n^2 &= n^3 - n + 3\Sigma n\\ &= n(n+1)(n-1) + \frac{3n(n+1)}{2}\\ &= \frac{n(n+1)(2n+1)}{2}.\end{aligned}$$

Therefore

$$\boxed{\Sigma n^2 = \frac{n(n+1)(2n+1)}{6},} \tag{16}$$

which gives the sum of the squares of the first n natural numbers.

50.2. The sum of the cubes of the first n natural numbers, or Σn^3, is found by proceeding in a similar manner with the identity

$$x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1.$$

In this case, after performing the addition as in the preceding case, substituting for Σn^2 and Σn, and simplifying, we get

$$\boxed{\Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.} \tag{17}$$

The details of the proof are left as an exercise.

Corollary. From formulas (15) and (17) we observe that

$$\Sigma n^3 = \{\Sigma n\}^2.$$

51. Series related to Σn, Σn^2, Σn^3. Formulas (15), (16), and (17) may be used in the summation of certain series formed from the natural numbers.

EXAMPLE. Find the sum of n terms of the series

$$1\cdot5+4\cdot7+7\cdot9+\cdots\ .$$

The terms of this series are formed by multiplying corresponding terms of the two arithmetic series 1, 4, 7, $\cdots$ and 5, 7, 9, $\cdots$. Hence, the nth term is the product of two factors, one being the nth term of the first arithmetic series, the other being the nth term of the second.

Therefore,

$$(a)\qquad t_n=(3n-2)(2n+3)=6n^2+5n-6.$$

Thus we have:

$$\begin{array}{llll}
t_1 & =6\cdot1^2 & +\ 5\cdot1 & -\ 6, \\
t_2 & =6\cdot2^2 & +\ 5\cdot2 & -\ 6, \\
t_3 & =6\cdot3^2 & +\ 5\cdot3 & -\ 6, \\
\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
t_{n-1} & =6(n-1)^2 & +\ 5(n-1) & -\ 6, \\
t_n & =6n^2 & +\ 5n & -\ 6.
\end{array}$$

By addition, we obtain

$$(b)\qquad s_n\ =6\Sigma n^2\ +5\Sigma n\ -6n.$$

Substitution from formulas (15) and (16) gives, after simplification,

$$s_n\ =\frac{n}{2}(4n^2+11n-5).$$

51.1. If, in the preceding example, we compare the expression (a) for t_n with that given by (b) for s_n, an interesting fact appears. The latter expression is seen to be obtained from the former by inserting the symbol Σ before each power of n and multiplying the constant term by n. The proof of this fact shows that this way of *expressing* s_n applies whenever t_n is a polynomial in n; moreover, it furnishes a method of *finding* s_n provided that t_n is a linear, quadratic, or cubic function of n, since Σn, Σn^2, and Σn^3 are known. For example, the following rule can be proved as in the foregoing example; the details are left as an exercise.

If the nth term of a series is
an^3+bn^2+cn+d,
then the sum of n terms is
$a\,\Sigma n^3+b\,\Sigma n^2+c\Sigma n+dn$.

EXERCISES

1. Find the sum of n terms of the following series:

(*a*) $3\cdot 5+5\cdot 9+7\cdot 13+\cdots$,

(*b*) $2\cdot 2+3\cdot 5+4\cdot 8+\cdots$,

(*c*) $4^2+7^2+10^2+\cdots$.

In each case, check the answer by letting $n=1$ and $n=2$.

2. Evaluate $1\cdot 3+2\cdot 5+3\cdot 7+\cdots+20\cdot 41$.

3. If the nth term of a series is given by the function $2n^2-5n$, what is the sum of n terms? Find the first two terms in the series and check your result. What is the sum of 7 terms?

4. The pth term of a series is p^2-3p. How many terms of this series must be taken to give the sum 0?

5. What is the value of $41^2+42^2+43^2+\cdots+50^2$?

EXERCISES ON CHAPTER VI

A

1. Find the sum of each of the following series, where the sum exists:

(*a*) $3+0-3-\cdots$ to 15 terms;

(*b*) $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{7}$;

(*c*) $4+3+9/4+\cdots$ to infinity;

(*d*) $1\cdot 5+2\cdot 6+3\cdot 7+\cdots$ to 10 terms;

(*e*) $150+15+1.5+\cdots$ to 8 terms;

(*f*) $9+15+25+\cdots$ to infinity.

2. Write down the 12th and the *p*th term of each of the following series:

(*a*) $20, -40, 80, \ldots$;

(*b*) $10, 12, 15, 20, \ldots$;

(*c*) $u+3v, u, u-3v, \ldots$;

(*d*) $2\cdot 3+5\cdot 7+8\cdot 11+\cdots$.

3. Counting parents, grand-parents, great-grand-parents, etc., how many ancestors does one have if he goes back 15 generations?

4. A number of logs are piled together in layers. The bottom layer consists of 20 logs laid on the ground side by side; the second layer consists of 19 logs placed on top of these; the third layer consists of 18 logs; and so on. If the top layer has 7 logs, how many logs are there in the pile? How many more logs could be added to the pile by adding more layers in this way?

5. Express the recurring decimal $.0191919\cdots$ as a vulgar fraction.

6. Find the arithmetic, geometric, and harmonic means of -8 and -18.

7. A small culture of bacteria is prepared on a glass plate. If the area covered by bacteria is doubled every hour and if the plate is half covered in a day, when was it one-quarter covered? When was it one-sixteenth covered?

8. If the nth term of a series is given by $6n^2+2n$, find the sum of n terms.

9. What number must be added to each of the numbers 1, 5, 13 so that the sums will form a geometric series?

10. If the first term of a geometric series is a^{-10}, and the common ratio, a^2, find t_{21} and t_n.

11. What is the value of $20+28+36+\cdots+140$?

12. The first term of an arithmetic progression is 45 and the 20th term is 102. Find d.

B

13. The sum of the first three terms of an arithmetic series is 24 and the fourth term is 14. Find the series.

(Hint: Let the first three terms be $a-d$, a, $a+d$.)

14. Find the 40th term and the sum of 40 terms of the series $7\cdot 15+10\cdot 22+13\cdot 29+\cdots$.

15. A squad of 40 soldiers is lined up for drill 3 feet apart facing north on an east-and-west line. They receive the order *quick march* and then the order that each should halt when he reaches a marked line which stretches north-east. The man on the west end of the line marched 100 yards before reaching the designated line. Find the sum of the distances marched by all the men.

16. Prove that, for any geometric series, $s_1=(r^2+1)s_2$.

17. Plot the function $y=2^x$, giving x the values 0, 1, 2, 3, $\cdots$. Draw the ordinates corresponding to these values of x. Show that these ordinates represent the successive terms of a geometric series.

18. The first term of a harmonic series is $\frac{1}{4}$ and the sum of the first three terms is $\frac{13}{24}$. Find the series.

19. The sum of three numbers in geometric progression is 65, and the third number is 9 times the first. What are the numbers?

20. Prove that if k is the geometric mean of m and n, the equation $mx^2+2kx+n=0$ has equal roots.

21. The mth term of a series is given by the function $7m+2$. Investigate the type of the series and find the sum of t terms.

22. A tank contains 100 gallons of alcohol which is diluted by drawing off 10 gallons and replacing it with water. Later 10 gallons of the mixture are drawn off and replaced with 10 gallons of water. This is repeated until the operation has been performed 6 times. If the liquid is thoroughly mixed after each withdrawal, how many gallons of alcohol are left in the tank?

23. Show that the roots of the equation $x^2-2Ax+G^2=0$ are two numbers of which A is the arithmetic mean and G is a geometric mean.

24. If the numbers $m+n$, $2n$, $n+k$ are in harmonic progression, show that m, n, k are in geometric progression.

25. If, in a geometric series, $s_n=A$ and $s_{2n}=B$, find r.

26. If p^2, q^2, r^2 are in arithmetic progression, show that $q+r$, $r+p$, $p+q$ are in harmonic progression.

27. A pile of ties is placed on a railroad track under construction. A man starts from the pile and carries the ties, one at a time, distributing them at intervals of 2 feet along the track. If he places the first tie 12 feet from the pile and the others farther away, how far has he walked when he has placed 25 ties and returned to the pile?

28. The qth term of a series is given by the function 3^q-2q. Find the sum of r terms of this series.

29. Use formulas (11) and (13) to obtain s_n as a function of a, l, and r.

30. Work out the details of the exercise suggested in sec. 51.1.

31. The sum of four numbers in geometric progression is 30 and the arithmetic mean between the second and third numbers is 6. Find the numbers.

32. If the numbers of the arithmetic series 2, 5, 8, 11, · · · are written in order in a book at the uniform rate of 100 to a page, beginning with page one, find (*a*) on what page (if any) the number 11,111 occurs, (*b*) the sum of all the numbers on the first 400 pages.

33. Prove that if an arithmetic series has an odd number of terms, its sum equals the product of the middle term multiplied by the number of terms.

34. The mid-points of the sides of an equilateral triangle are joined, forming another equilateral triangle inscribed within the first. The mid-points of this second equilateral triangle are joined, forming a third equilateral triangle, and so on indefinitely. Show that the sum of the perimeters of all the inscribed triangles is equal to the perimeter of the original triangle. Also, prove that the sum of the areas of all the inscribed equilateral triangles is 1/3 of the area of the original triangle.

C

35. If the arithmetic, geometric, and harmonic means between two positive, unequal numbers p and q are denoted respectively by A, G, and H, prove that A, G, and H form a geometric progression. Show also that $A > G > H$.

36. The harmonic mean of 1 and x is given by (9) as $h = \frac{2x}{1+x}$. How does h vary as x takes on very large positive values? As x approaches -1 from higher values? As x approaches -1 from lower values? As x takes on very small negative values? By making use of this information and plotting a number of points, draw a graph to show the relationship between h and x in this formula.

37. A square S_1 is inscribed within a circle C_1 and the part of the circle outside the square is coloured black. A circle C_2 is inscribed within S_1 and a second square S_2 is inscribed in C_2. As before, the part of C_2 lying outside of S_2 is coloured black. This process of inscribing squares and circles and colouring outlying regions is to be kept up indefinitely. Prove that the areas of the regions coloured in the successive steps of the process form a geometric series and find the total fraction of the original circle coloured black. If the radius of the original circle is k, find the sum of the perimeters of all the circles inscribed in this process. Find also the sum of the perimeters of all the squares.

38. Find the sum of n terms of the series

$$1\cdot 4\cdot 8+2\cdot 6\cdot 11+3\cdot 8\cdot 14+\cdots.$$

39. Assuming the formulas for Σn and Σn^2, find the sum of the squares of the first n terms of the general arithmetic progression.

40. An arithmetic, a geometric, and a harmonic series each have the same first term, the same last term, and the same number of terms, namely $2n-1$. Prove that the middle term of the geometric series is the geometric mean of the rth term of the arithmetic series and the $(2n-r)$th term of the harmonic series.

41. In a geometric progression, the first term is a, and the nth term is l. Show that the pth term is equal to $(a^{n-p}\, l^{p-1})^{\frac{1}{n-1}}$.

42. Find the sum of the series $1+3x+5x^2+7x^3+\cdots$ to n terms and to infinity. $(-1<x<1)$.

Hint: Multiply each term of this series by x and subtract the new series from the old as in section 48.

43. Prove that the infinite product $4^{\frac{1}{4}}\times 16^{\frac{1}{16}}\times 64^{\frac{1}{64}}\times\cdots$ equals $4^{\frac{4}{9}}$.

44. Find the sum of k terms of the series formed by taking the 1st, 2nd, 4th, 8th, 16th, $\cdots$ terms of the arithmetic series $p+(p+q)+(p+2q)+(p+3q)+\cdots$.

45. In an arithmetic series, the square of the k^2th term is equal to the sum of the first k^2 terms multiplied by twice the common difference d. Find the first term in terms of k and d.

46. If S denotes the sum of n terms, P the product of n terms, and R the sum of the reciprocals of n terms in any geometric series, prove the relation $(S/R)^n=P^2$.

47. Prove that the sum of 100 terms of the series

$5+5.5+5.55+5.555+\cdots$ is approximately $554\frac{76}{81}$.

48. What condition must a, b, c satisfy in order that the roots of $x^3+ax^2+bx+c=0$ may be in arithmetic progression?

49. The sum of three numbers in geometric progression is 13 and the sum of their squares is 91. Find the numbers.

50. Find the sum of n terms of the series

$$\frac{a}{a}+\frac{a+d}{ar}+\frac{a+2d}{ar^2}+\frac{a+3d}{ar^3}+\cdots.$$

CHAPTER VII

PERMUTATIONS AND COMBINATIONS

52. Permutations. The problem of finding the number of ways of arranging a given set of objects and problems related to it have aroused the curiosity and called forth the ingenuity of men for many generations. In how many ways can ten boys be formed into a line? How many automobile licence plates with four-digit numbers can be made using the digits

1, 2, 3, 4, 5, 6, 7, 8, 9?

In how many ways can one take a walk of 7 blocks, if he goes 4 blocks east and 3 blocks north? The investigation of such problems is the object of the first part of this chapter.

DEFINITION. *A permutation of a number of things is any arrangement of these in a definite order.* For example, 123, 231, and 321 are permutations of the three digits 1, 2, and 3. In this case, the digits have been arranged three at a time, or all together. It is also possible to arrange them two at a time or one at a time; the permutations would then be 12, 13, 21, 23, 31, 32 in the first case, and 1, 2, 3 in the second.

Before reading further, the student is advised to carry out the following experiment:

> Write down the permutations, all together, of the letters *abc* and also of the letters *abcd*. Write the permutations of the letters *abc* one at a time and two at a time. Write the permutations of the letters *abcd* one at a time, two at a time, and three at a time. The object of the experiment is to discover

ways of calculating, without writing the arrangements down, the number of permutations of the five letters *abcde* taken one, two, three, four, or five at a time.

52.1. A typical problem in permutations is provided when books are arranged on a shelf. Suppose that three books, an algebra, (A), a geometry, (G), and a trigonometry, (T), are placed on a bookshelf on which are three available spaces. In how many ways can the three books be arranged in the three spaces? One way of arriving at a solution is to write down all the different ways of assigning the three books to the three places thus:

AGT	GTA	TAG
ATG	GAT	TGA.

We have six permutations or arrangements of the three books taken three at a time. Such a procedure is always possible, but it is likely to become laborious. The following reasoning offers a more convenient solution for such problems.

There are three spaces to fill. In the first space, we can put A or G or T so that the first space can be filled in 3 ways. For each of these 3 ways of filling the first space, there are 2 ways of filling the second space, since either of the two remaining books can be used. Therefore, the first two spaces can be filled in 3×2 or 6 ways. After the first two spaces have been filled in any one of these 6 ways, there is just one way of filling the third space, since only one book remains. Hence, the three spaces can be filled in 6×1 or 6 ways. A convenient method of recording the result is to represent the spaces by dashes with a number above each to indicate the number of ways in which the space can be filled, thus:

$$\underline{3}\cdot\underline{2}\cdot\underline{1}=6.$$

Consider next a variation of this problem in which there are available at least three books of each kind. In this case, as before, the first space can be filled with A or G or T and hence in 3 ways. After it is filled in any one of these ways, there remain books of the three kinds so that the second space can also be filled in 3 ways. Similarly, the third space can be filled in 3 ways. Therefore, the number of ways of filling all three spaces is

$$\underline{3}\cdot\underline{3}\cdot\underline{3}=27.$$

This problem differs from the preceding in that the choice of a given kind of book may be repeated because there are at least three of each kind. The 27 arrangements are here indicated:

AAA, AAG, AAT, AGA, ATA, AGG, ATT, AGT, ATG,
GGG, GGA, GGT, GAG, GTG, GAA, GTT, GAT, GTA,
TTT, TTA, TTG, TAT, TGT, TAA, TGG, TAG, TGA.

In this case, the books have been arranged three at a time *with repetitions allowed.* It is of first importance to observe that the *order* of the objects is the significant feature of a permutation: *a change in order gives a different permutation.*

The reasoning used in the foregoing problems provides a general method for handling problems in permutations.

53. Fundamental principle. *If one operation can be performed in m ways, and, after it is performed in any one of these ways, a second operation can be performed in n ways, then the two operations can be performed together in mn ways.*

It is given that for each way of performing the first operation, there are n ways of performing the second. Therefore, for m ways of performing the first, there are $m\times n$ ways of performing the two operations. The reasoning here used may be extended to any number of operations. The general result is stated in the following corollary.

53.1. COROLLARY. *If an operation can be performed in m_1 ways, and, after it is performed in any one of these ways, a second* *operation can be performed in m_2 ways, and, after it is performed in any one of these ways, a third operation can be performed in m_3 ways and so on for k operations, then the k operations can be performed together in*

$$m_1 \cdot m_2 \cdot m_3 \cdots m_k \text{ ways.}$$

53.2. In working problems, the student is advised to form the habit of giving the reasoning in each problem independently. He should not expect all problems to yield to a formula; a special restriction in a problem may necessitate special treatment. The important thing is to have a clear grasp of the fundamental principle. The application of this principle is illustrated in the following example.

EXAMPLE. Given the digits 1, 2, 3, 4, and 5, find how many four-digit numbers can be formed from them (*a*) if no digit may be repeated, (*b*) if repetitions of a digit are allowed, and (*c*) if the number must be odd, with repetitions allowed.

(*a*) There are four places to fill. The first place can be filled with any one of the five digits and so in 5 ways. Then, since no digit may be used twice, there remain four digits, any one of which may fill the second place. Thus the second place can be filled in 4 ways, and similarly the third in 3 ways, and the fourth in 2 ways. From the fundamental principle, it follows that the number of four-digit numbers is given by

$$\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} = 120.$$

(*b*) Here, the use of a digit may be repeated so that after the first place has been filled with any one of the five digits, there are still 5 ways of filling the second place and so on. Therefore, the number of numbers in this case is

$$\underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} = 625.$$

(*c*) Since the number must be odd, the final digit has to be 1 or 3 or 5. Therefore, the fourth place can be filled in just

3 ways. After this is done in any one of these ways, the remaining spaces can each be filled in 5 ways because repetitions are allowed as in part (*b*). The number of numbers is given by

$$\underline{5}\cdot\underline{5}\cdot\underline{5}\cdot\underline{3} = 375.$$

Note: If some operation must be performed in a special way, it is generally advisable to do it *first.*

EXERCISES

1. In how many ways can 8 people line up at a theatre box-office?

2. How many five-digit numbers can be formed from the integers 1, 2, 4, 6, 7, 8, if no integer can be used more than once? How many of these numbers will be even? How many odd?

3. If the call letters of Canadian broadcasting stations must begin with the letter *C*, how many different stations could be designated using 3 letters with repetitions of a letter allowed? How many, using 4 letters without repetitions?

4. In how many ways can 3 letters be posted in 6 postboxes, if each letter must be posted in a different box? If the letters are not necessarily posted in different boxes, how many ways are there of posting them?

5. There are 7 seats available in a sedan. In how many ways can 7 persons be seated for a journey if 3 are able to drive?

6. A passenger train has 9 coaches. In how many ways can 4 people ride in the train if they must ride in different coaches?

7. In how many ways can 5 students be seated in a classroom with 20 desks?

8. Twelve boys try out for the hockey team. Two can play only in goal, 4 only on defence, and the rest can play either centre or forward. In how many different ways could the coach arrange a team?

9. In how many ways can 5 boys and 5 girls be seated alternately in a row of 10 chairs, numbered from 1 to 10, if a boy always occupies the first chair?

10. In how many ways can 3 different parcels, A, B, and C, be given to any 3 of 15 persons? If a specified person must receive A, and if no person is to receive more than one parcel, in how many ways can the parcels be distributed?

11. An encylcopedia consists of 9 volumes numbered 1 to 9. In how many ways can the 9 volumes be arranged together on a shelf so that some or all of the volumes are out of order?

12. How many five-digit numbers can be formed? How many of these begin with 2 and end with 4? How many do not contain the integer 5? How many are divisible by 5?

13. Five boys compete in a race. In how many ways can the first 2 places be taken?

54. **Definitions.** (1) *An arrangement of r things, taken from a group of n different things, is called a* ***permutation of the n things, r at a time.*** The total number of such permutations is denoted by ${}_nP_r$, where $r \leqq n$.

(2) A convenient notation for representing a product of consecutive integers is provided by the factorial symbol, written $n!$ or $\underline{|n}$, and read: "factorial n". This symbol is defined as follows, where n is any positive integer:

$$n! = n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1 \quad .$$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$

55. Formulas in permutations. THEOREM. *The number of permutations of n different things, taken r at a time, without repetitions, is*

$${}_nP_r=\frac{n!}{(n-r)!}.$$

The proof is merely an application of the fundamental principle. Suppose that we have r spaces to fill and n things from which to choose. The first space can be filled with any one of the n things and so in n ways. After this is done in any one of these ways, there are $n-1$ things left, any one of which may be put in the second space. Therefore, the second space can be filled in $n-1$ ways. When this has been done in any one of these ways, the third space can be filled in $n-2$ ways, and so on. In each of these numbers, the integer subtracted from n is one less than the number of the space. Hence, the tenth space can be filled in $n-9$ ways, and the last or rth space in $n-(r-1)$ or $n-r+1$ ways. From the corollary of sec. 53.1, it follows that the number of ways in which the r spaces can be filled is

$$(1)\qquad {}_nP_r=n(n-1)(n-2)\cdots(n-r+1).$$

This formula, consisting of r factors, takes a more convenient form if its numerator and denominator are each multiplied by $(n-r)!$. The numerator then becomes $n!$ and the formula is

$$(2)\qquad \boxed{{}_nP_r=\frac{n!}{(n-r)!}.}$$

Thus, ${}_nP_r$ is a function of the two variables n and r, defined for positive, integral values of n and r such that $r<n$.

55.1. COROLLARY. *The number of permutations of n different things taken n at a time, or all together, is* $n!$.

This follows at once from (1) by setting $r=n$. By defining $0!$ to be 1, we make the formula (2) hold also in this case.

55.2. THEOREM. *The number of permutations of n different things taken r at a time, where repetitions are allowed, is* n^r.

Suppose, as before, that we have r spaces to fill. Since each of the n things may be used as often as desired, there are n ways of filling the first space and also n ways of filling each of the remaining $r-1$ spaces. Hence, from the fundamental principle, the total number of arrangements is the product

$$n \cdot n \cdot n \cdot n \cdot \; \cdot \; \cdot \text{ to } r \text{ factors, or } n^r.$$

56. Circular permutations. Four boys can be arranged in a line in 4! ways. The problem may be different if they are arranged about a circle or other closed curve. Consider a circular arrangement (fig. 24a) in which only their positions with respect to each other are important. If each boy moves one place to the left, the relative arrangement (fig. 24b) is not affected since each boy has the same right-hand and left-hand neighbours as before. Such arrangements give rise to *circular permutations.*

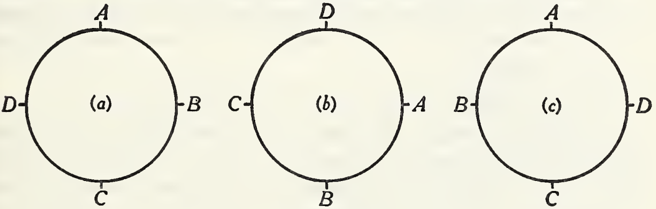


FIG. 24

The arrangement in fig. 24c is that of fig. 24a in opposite order. It counts as a different circular permutation, since any boy has different right-hand and left-hand neighbours from those he had before.

In circular permutations, it is convenient to assign one of the objects to a position in the ring and then place the others with reference to it. In this example, after one boy is placed in the ring, the three remaining places may be numbered from his left so that arranging three boys in these places is the same as arranging them in line. Hence, the other three boys can be arranged with respect to the first in 3! ways. Similarly, n boys can be arranged in a ring in $(n-1)!$ ways.

An interesting variation occurs when beads are strung on a ring. Here, the arrangements of fig. 24a and fig. 24c are really the same string of beads, since one of them is merely the other turned over. Therefore, such pairs of arrangments in opposite order count as one stringing so that n beads can be arranged on a ring in $\frac{(n-1)!}{2}$ different ways.

57. Examples. The student is advised to attempt each of the following examples himself, before reading the solution given.

EXAMPLE 1. How many words* can be formed from the letters of the word *hyperbola*, taken all together? In how many of these words will the letters *hy* occur together? In how many will *hy* not occur together?

From the corollary of sec. 55.1, the number of ways of arranging 9 different letters all together is $9! = 362,880$. Thus, the required number of words is 9! if there are no restrictions. If the letters *hy* are to occur together, it is convenient to think of them as one letter. We now have 8 different letters to be arranged 8 at a time; this gives 8! arrangements. In each of these arrangements, the order *hy* may be changed to *yh*. Hence, the total number of words in which the letters *hy* occur together is $2(8!) = 80,640$. The difference,

$$362,880 - 80,640 = 282,240,$$

is the number of words in which these letters do *not* occur together.

---

*A "word", in this sense, means any arrangement of letters.

EXAMPLE 2. If there are two railroads from Toronto to Montreal and four steamship lines operating from Montreal to Southampton, in how many ways is it possible to go from Toronto to Southampton and return, via Montreal, without travelling over any line twice?

The trip from Toronto to Montreal can be made in 2 ways; after it is made in one of these ways, the trip from Montreal to Southampton can be made in 4 ways; then, the return from Southampton to Montreal can be made in 3 ways, since a different steamship line must be used; finally, there is only 1 way of returning from Montreal to Toronto. Therefore, by the fundamental principle, the number of ways of making the entire journey is

$$2 \cdot 4 \cdot 3 \cdot 1 = 24.$$

EXAMPLE 3. (*a*) How many automobile licence plates bearing five-digit numbers can be made if no licence number starts with 0? (*b*) If letters of the alphabet are used in place of the first two digits and the third digit is not 0, how many plates can be made?

(*a*) The first digit may be any one of the integers 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, there are 9 ways of choosing it. For the second digit, we can use any one of 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, so that there are 10 ways of choosing the second digit. Similarly, there are 10 ways of choosing each of the remaining digits, since the use of a digit may be repeated. By the fundamental principle, the number of licence plates is

$$\underline{9} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 90{,}000.$$

(b) Reasoning as in part (*a*), we find the number of plates to be

$$\underline{26} \cdot \underline{26} \cdot \underline{9} \cdot \underline{10} \cdot \underline{10} = 608{,}400.$$

EXAMPLE 4. A musical concert is to consist of three songs and two violin selections. In how many ways can the programme be arranged so that the concert begins and ends with a song, and neither violin selection comes immediately after the other?

> There are five spaces on the programme to be filled. The first space can be filled in 3 ways, since any one of the three songs can be used; the fifth space can be filled in 2 ways because either of the two remaining songs can be used. A violin selection must occur second on the programme, otherwise the two violin numbers would follow in immediate succession; therefore, the second space can be filled in 2 ways, the third in 1 way, and the fourth in 1 way. Thus the number of ways of arranging the programme is
>
> $$\underline{3}\cdot\underline{2}\cdot\underline{1}\cdot\underline{1}\cdot\underline{2}=12.$$

EXERCISES

1. Evaluate the following: ${}_9P_8$, ${}_{12}P_5$, ${}_mP_2$, ${}_7P_7$.

2. If ${}_nP_2=56$, find the value of n.

3. How many words can be formed from the letters of the word *fragments* (*a*) taken all at a time, (*b*) taken 8 at a time, (*c*) taken 4 at a time?

4. A student has 6 examinations to write and there are 10 examination periods available. How many possible arrangements are there of his examination programme?

5. In how many ways can 10 boys form a ring, (*a*) if relative position only is considered, (*b*) if relative position only is considered and, in addition, a certain pair must be together?

6. In how many ways can 5 different keys be put on a key ring?

7. Given four flags of different colours, how many different signals can be made by arranging them on a vertical mast, if at least two flags must be used for each signal?

8. Prove that the number of three-letter words that can be formed from the letters of the word *background* is the same as the number of words that can be made by rearranging the letters in *ground.*

9. A passenger train consists of two baggage cars, four day coaches, and three parlour cars. In how many ways can the train be made up if the two baggage cars must come in front, and the three parlour cars must come in the rear?

10. In how many ways can 7 books be arranged on a shelf, (*a*) if two specified books must always be side by side, (*b*) if these two books must not be side by side?

11. In geometry, rectilineal figures are commonly labelled by placing letters at their vertices. How many ways are there of labelling a triangle with letters of the alphabet? How many ways are there of labelling a pentagon?

12. How many five-letter words can be formed from 10 different letters (*a*) with repetitions, (*b*) without repetitions? In how many of the words of (*a*) will there actually occur repeated letters?

13. A round table has a special type of chair at its head. In how many ways can 6 people arrange themselves at it?

14. How many three-letter words can you make from the letters of your name, if the words must begin and end with different consonants and have a vowel in the middle?

58. Combinations. We have seen that the number of arrangements of three different books, two at a time, on a shelf is ${}_8P_2$ or 6. In these arrangements, the order in which the books occur on the shelf is taken into account. A different problem arises when two books are to be chosen from three different books, A, B, and C, without taking order into account. There are then only three possible selections, namely:

$$AB,\ BC,\ CA.$$

The selection AB is the same as BA since the order of the books is immaterial. Such a selection is called a ***combination*** of three things, two at a time. This example makes clear the difference between a permutation and a combination. *In a permutation, order is taken into account; in a combination, it is not.* Each of the foregoing selections or combinations gives rise to two permutations. For example, the combination AB gives the two permutations AB and BA. Hence, the three combinations yield the six possible permutations.

DEFINITIONS. *A* ***combination*** *is a* ***selection*** *of objects, considered without regard to their order. A set of r objects selected, without regard to their order, from a group of n different objects is called a* ***combination of the n objects, r at a time.*** The total number of such combinations is denoted by ${}_nC_r$, where $r \leqq n$.

58.1. THEOREM. *The number of combinations of n different things taken r at a time, without repetitions, is given by the formula*

$${}_nC_r = \frac{n!}{r!\,(n-r)!}.$$

Each combination of r things gives rise to $r!$ permutations by changing the order of the things in the combination. Therefore, ${}_nC_r$ combinations yield ${}_nC_r \times r!$ permutations which are all different. Also, since each permutation of r things arises from some combination, all possible permutations of the n things, r at a time, are obtained in this way. Hence

$${}_nC_r \times r! = {}_nP_r = \frac{n!}{(n-r)!},$$

or

$$\boxed{{}_nC_r = \frac{n!}{r!(n-r)!}.} \tag{3}$$

Thus, ${}_nC_r$ is a function of both n and r. The nature of its dependence on these variables will be illustrated in sec. 63.2.

COROLLARY. From (3), it follows at once that

$$\boxed{{}_nC_{n-r} = \frac{n!}{(n-r)!r!} = {}_nC_r .} \tag{4}$$

This result might have been inferred directly. Whenever a selection of r things is made from n, a selection of $n-r$ things is left behind. The number of selections of r things is therefore the same as that of $n-r$ things.

59. Examples. In some problems, it is easy to decide from the wording whether permutations or combinations are involved; in others, this question presents more difficulty. If, from the nature of the problem, the order of the things in a set taken must be considered, then the problem is one of permutations. If the order in the sets taken is immaterial, the problem concerns combinations. The following examples illustrate methods of solution.

EXAMPLE 1. In how many ways can a selection of 3 books be made from a library shelf containing 100 books?

The number of ways of selecting 3 books from 100 is given by formula (3) to be

$${}_{100}C_3 = \frac{100!}{3!\,97!} = \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3} = 161{,}700.$$

EXAMPLE 2. Prove that ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.

${}_{n+1}C_r$ is the number of ways in which a selection of r objects can be made from $n+1$ different objects, without restrictions. If a specified object is not to be included in the selection, the r objects must be selected from the n remaining objects and this can be done in ${}_nC_r$ ways. If the specified object must be included, there are $r-1$ objects to be selected from n and this can be done in ${}_nC_{r-1}$ ways. The total number of selections is obtained by adding the number in which the specified object occurs to the number in which it does not occur, since no other cases are possible. Therefore, the total number is

$${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r.$$

Example 3. In how many ways can a committee of three be chosen from four married couples (*a*) if all are equally eligible, (*b*) if the committee must consist of two women and one man, (*c*) if a husband and wife cannot both serve on the same committee?

(*a*) In a committee, the arrangement is of no importance so that the problem is that of selecting 3 people from 8 in all possible ways. From (3), the total number is

$${}_8C_3 = \frac{8!}{3!\,5!} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56.$$

(*b*) The 2 women can be selected in ${}_4C_2$ or 6 ways. One man can be selected in ${}_4C_1$ or 4 ways. Hence, by the fundamental principle, the number of ways of selecting 2 women and 1 man is

$${}_4C_2 \times {}_4C_1 = 6\times 4 = 24.$$

(*c*) Since a husband and wife cannot both serve, therefore three couples must be represented on the committee. Three couples can be chosen from four in ${}_4C_3 = 4$ ways. When three couples have been selected, 2 choices can be made from the first (either husband or wife), 2 from the second, and 2 from the third. By the fundamental principle, the total number of choices for the committee is

$$4\cdot 2\cdot 2\cdot 2 = 32.$$

Example 4. Twenty persons are to travel in a bus which can carry 12 passengers inside and 8 outside. If four of the persons will not travel inside, and five will not travel outside, in how many ways can the passengers be seated (*a*) if the arrangement of the passengers inside, or outside, is not considered, (*b*) if the arrangement inside and outside is considered?

(*a*) First, let the 5 persons who must sit inside and the 4 persons who must sit outside take seats. This leaves 11 persons to occupy the remaining 11 seats, 7 of which are inside and 4 outside. Since the arrangement of passengers is not to be considered, 7 persons can be selected for the inside seats in ${}_{11}C_7$ ways. Only 4 persons remain, so that there is just 1 way of selecting 4 outside passengers, after the inside seats are occupied. Therefore, by the fundamental principle, the number of ways of seating the passengers is

$$ {}_{11}C_7 \times 1 = \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = 330. $$

(*b*) The 12 passengers inside the bus can be arranged in $12!$ ways; after they have been arranged in any one of these ways, those outside can be arranged in $8!$ ways. Thus, by the fundamental principle, each choice of seating in part (*a*) will yield $12! \times 8!$ different choices, if the arrangement inside and outside is considered. In this case, the total number is

$$ 330 \times 12! \times 8!\,. $$

This method of making selections first, and then finding the number of arrangements arising from each selection, is a useful one.

EXERCISES

1. Evaluate the following: ${}_9C_3$, ${}_9C_6$, ${}_kC_3$, ${}_5C_5$.

2. Solve the following equations for n:

(*a*) ${}_nC_2 = 45$. (*b*) $\dfrac{{}_nP_4}{{}_{n-1}C_3} = 60$. (*c*) ${}_nC_8 = {}_nC_{12}$.

3. In how many ways can a committee of 3 be chosen from 7 people?

4. A contractor needs 4 carpenters and 10 apply for the jobs. In how many ways can he pick out four?

5. Ten points are taken on the circumference of a circle. How many chords can be drawn by joining them in all possible ways? With these points as vertices, how many triangles can be drawn? How many quadrilaterals? How many hexagons?

6. In how many ways can a selection of 4 books be made from 9? If a certain book must be taken, in how many ways can this selection be made? In how many ways can it be made if a certain book must be left?

7. A company of 20 men is to be divided into 3 sections so that there are 3 men in the first, 5 in the second, and 12 in the third. In how many ways can this be done?

8. Write down a symbol for the number of combinations of 20 things, taken 4 at a time, and for the number of combinations of 100 things, taken 98 at a time. Compute the numerical value of each symbol, and find which is the greater.

9. A pack of playing cards used in bridge consists of 52 different cards. If a hand is made up of 13 cards, use the factorial notation to express the number of possible hands.

10. A pack of cards is made up of 13 spades, 13 hearts, 13 diamonds, and 13 clubs. How many different hands can be formed each containing 5 spades, 4 hearts, 2 diamonds, and 2 clubs?

11. Six candidates contest an election for two similar offices. If a voter may mark his ballot for one or for two candidates, in how many ways can he cast his vote?

12. How many five-letter words, each consisting of 3 consonants and 2 vowels, can be formed from the letters of the word *equations*?

13. In how many ways can 4 persons be selected from 5 married couples (*a*) if the selection must consist of 2 women and 2 men, (*b*) if a husband and wife cannot both be selected?

EXERCISES ON PERMUTATIONS AND COMBINATIONS

A

1. In how many ways can 6 boys form (*a*) a row, (*b*) a ring, if relative position only is considered?

2. In how many ways can 10 beads of different colours be strung to form a necklace?

3. Find the number of ways in which a man can choose 3 gifts from 10 different articles.

4. A railway has 50 stations. If the names of the point of departure and the destination are printed on each ticket, how many different kinds of single tickets must be printed?

5. In how many ways can 15 different objects be divided among A, B, and C, if A is to receive 2 objects, B is to receive 3, and C is to receive 10?

6. Given 20 points, no 3 of which are in a straight line, find the number of straight lines that can be drawn by joining pairs of these points.

7. A ring of 8 boys is to be enlarged by the addition of 5 girls. In how many ways can this be done if no two girls are to stand beside each other?

8. A town council is made up of a mayor and 6 aldermen. How many different committees of 4 can be formed, (*a*) if the mayor is on each committee, (*b*) if the mayor is on no committee?

9. How many four-letter words can be made from the letters of the word *zephyr*? How many of these words will not contain the letter *r*? How many will contain *r*? How many will begin with *z* and end with *r*?

10. In how many ways can a coach choose a team of 5 from 10 boys (*a*) if 2 specified boys must be included, (*b*) if there are no restrictions?

B

11. In how many numbers between 1000 and 9999 inclusive does the digit 3 occur?

12. How many words, each of 2 vowels and 2 consonants, can be formed from the letters of the word *involute*?

13. How many quadrilaterals can be formed, each having as its vertices 4 of the vertices of a given regular polygon of 20 sides, if no 2 of the selected 4 are opposite vertices of the given polygon?

14. Prove that ${}_nC_r+{}_nC_{r-1}={}_{n+1}C_r$ by using formula (3) to replace the symbols on the left, and simplifying.

15. There are 10 chairs in a row. In how many ways can 2 persons be seated? In how many of these ways will the 2 persons be sitting in adjacent chairs? In how many will they have at least one chair between them?

16. How many diagonals has a 20-sided polygon? How many sides has a polygon with 35 diagonals?

17. Four positions of a certain sort can be held by women only, 5 positions of another sort by men only, and 3 positions of a third sort by either men or women. Find the number of ways in which these positions can be filled from 18 applicants of whom 8 are women and 10 are men.

18. Fifteen points lie in a plane in such a way that 5 of the points are on one straight line and, apart from these, no 3 points are collinear. Find the total number of straight lines that can be obtained by joining pairs of the 15 points.

19. How many six-digit numbers can be formed from 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number has 3 odd and 3 even digits and no two digits alike?

20. In how many ways can one assign to 2 soldiers different three-digit numbers? In how many ways can this be done if the three-digit numbers are composed of even digits only (zero being considered even)?

21. In an examination, 10 papers are set of which 2 are in mathematics. In how many different orders can the papers be given provided only that the papers in mathematics are not consecutive?

22. A stamp collecter has 8 different Canadian stamps and 10 different English stamps. Find the number of ways in which he can select 3 Canadian stamps and 3 English stamps and arrange them in 6 numbered spaces in his stamp album.

23. A symphony is recorded on 4 discs, both sides of each disc being used. In how many ways can the 8 sides be played on a phonograph so that some part of the symphony is played out of its correct order?

24. A railway coach has 10 seats facing backwards and 10 facing forwards. In how many ways can 8 passengers be seated, if 2 refuse to ride facing forwards and 3 refuse to ride facing backwards?

25. Given 10 beads of different colours, how many necklaces, each of which has 8 beads, can be strung from them, if the order of the beads on a necklace is taken into account?

26. From a company of 20 soldiers, a squad of 3 men is chosen each night. For how many consecutive nights could a squad go on duty without any two of the squads being identical? In how many of these squads would a given soldier serve?

60. Permutations of things which are not all different. In sections 52 to 57, we considered arrangements of things which were all different; each thing could be distinguished from the others. If some of these objects are indistinguishable, the number of permutations is, in general, smaller. For example, the integers 1, 2, 3 can be formed into $3!$ or 6 three-digit numbers, but the integers 3, 3, 3 yield only one three-digit number. We shall now investigate the number of arrangements of n things when some of these things are alike.

60.1. Consider the problem of arranging the letters of the word *bubble*, all together. Let the number of permutations be P. If in any arrangement such as

$$e\ b\ b\ u\ l\ b,$$

the three b's are replaced by b_1, b_2, b_3, this arrangement will give rise to $3!$ arrangements by permuting the b's without disturbing the other letters. In this way, each of the P permutations yields $3!$ permutations so that the total number becomes $P \times 3!$. As the six letters of the word are now all different, this total is the number of permutations of 6 different letters, all together. Hence,

$$P \times 3! = 6!$$

or

$$P = \frac{6!}{3!}.$$

Therefore, when three letters are alike, the number of permutations is $1/3!$ of the number obtained when all the letters are different. A similar method of reasoning was used to obtain ${}_nC_r$. The theorem which follows is a generalization of this problem.

60.2. THEOREM. *The number of permutations of n things, taken all together, when p of the things are alike and the rest different, is* $\frac{n!}{p!}$.

Let the number of arrangements be P. Replace the p similar things by p things, different from each other and from the rest. Then, each of the P arrangements gives rise to $p!$ arrangements by permuting the p things without disturbing the others. The number of arrangements has now become $P \times p!$, and since the n things are now all different,

$$P \times p! = n!$$

or

$$P = \frac{n!}{p!}.$$

Thus, when p things are alike, the number of permutations is $1/p!$ of the number when the things are all different. By repeated applications of this principle, we obtain the following corollary.

COROLLARY. *The number of permutations of n things, taken all together, when p of the things are alike of one kind, q alike of another, r alike of another, and so on, is*

$$\boxed{\frac{n!}{p!\,q!\,r!\ \cdots}.} \tag{5}$$

If, for any reason, the order of certain objects relative to one another is not to be changed, then these objects may not be permuted among themselves, and so for the purposes of the problem may be considered as identical. Hence, the above results apply to such problems. For example, if the digits 1, 2, 3 are to be kept in ascending order, the four-digit numbers which can be formed from 1, 2, 3, 4 are 1234, 1243, 1423, and 4123. This agrees with the fact that, if the 3 digits were identical, the number of arrangements of the 4 digits would be $\frac{4!}{3!} = 4$.

Example. (*a*) How many arrangements can be made from the letters of the word *independent*, taken all together? (*b*) Show that in 1/6 of these arrangements the letters *i*, *p*, *t* occur in the order *ipt*.

i
nnn
dd
eee
p
t

(*a*) The word independent consists of 11 letters, including 3 *e*'s, 3 *n*'s, 2 *d*'s, and the letters *i*, *p*, and *t*. Therefore, the number of permutations of these letters, taken all together, is, by (5),

$$\frac{11!}{3!\,3!\,2!}.$$

(*b*) If the letters *i*, *p*, *t* are always to occur in the order *ipt*, then these letters cannot be permuted among themselves. Hence we may treat them as if they were alike so that the number of permutations is now

$$\frac{11!}{3!\,3!\,3!\,2!}.$$

This is 1/6 of the number obtained in (*a*).

EXERCISES

1. Find the number of distinct arrangements of the letters of the word *committee*, using all the letters in each arrangement.

2. How many different numbers can be obtained by arranging the digits 2233344455, all together, in all possible ways?

3. How many permutations can be made using the letters of the word *institution*, all at a time? How many of these begin with *t* and end with *s*?

4. Find the number of ways in which 6 plus signs and 4 minus signs can be arranged in a row.

5. How many different numbers can be obtained by arranging the digits 123456789, all at a time, if the even digits must always remain in ascending order and the odd digits likewise?

6. Find the number of arrangements of the letters of the word *engineering*, taken all together. In how many of these are three *e*'s together? In how many are exactly two *e*'s together?

7. In how many ways can one take a walk, if he always walks 4 blocks east and 3 blocks north?

8. How many numbers greater than 3,000,000 can be formed from the digits 1, 1, 1, 2, 2, 3, 3?

61. Combinations of things which are not all different. A common type of problem involving combinations occurs when a selection, one or more at a time, is made from groups of similar things. For instance, in making a selection from three identical algebra books and two identical geometry books, there are 4 possibilities in selecting an algebra book; 1 or 2 or 3 or none of them may be taken. After the algebra books have been dealt with in any one of these 4 ways, the geometry books may be similarly dealt with in 3 ways. Hence, by the fundamental principle, there are 4×3 or 12 possibilities in the selection of books. This includes the case in which neither an algebra nor a geometry book is taken. As this case is to be ruled out, there are 11 ways left.

61.1. The foregoing example may be generalized and leads to the following theorem.

THEOREM. *The total number of selections which can be made from p things of one kind, q of another, r of another, and so on, is*

$$(p+1)(q+1)(r+1)\cdots-1. \tag{6}$$

The number of possibilities in making selections from the p things is $p+1$, since 1 or 2 or 3 or $\cdots$ or p or none of them may be taken. In the same way, the q things give rise to $q+1$ possibilities, the r things to $r+1$ possibilities, and so on. Therefore, by making use of the fundamental principle and excluding the case in which nothing is taken, we get the total number of selections to be

$$(p+1)(q+1)(r+1)\cdots -1.$$

61.2. Corollary. *The total number of selections which can be made from n different things is* $\mathbf{2^n-1}$.

This is obtained by setting $p=q=r=\cdots=1$ in formula (6). Or, since each of the n things may be taken or left, there are two ways of dealing with each thing and so 2^n-1 ways of making a selection of at least one thing from n.

Example 1. In how many ways can a selection of fruit be made from 5 oranges, 3 lemons, and 7 bananas?

Formula (6) gives the number of ways to be

$$6\cdot 4\cdot 8-1=191.$$

Example 2. From 6 of his friends, how many parties of one or more can a man invite to dinner?

It is a question of making a selection of at least one from 6 people. By using the corollary of sec. 61.2, we find the number of possible parties to be

$$2^6-1=63.$$

EXERCISES

1. Find the number of ways in which a selection of fruit can be made from 7 plums, 4 lemons, and 9 oranges.

2. In how many ways can a selection be made from 8 different books?

3. On stepping off a train, a man finds that he has a nickel, a dime, a quarter, and a half-dollar in his pocket. In how many ways can he give the porter a tip?

4. How many integral factors has the number 60, excluding unity and the number itself? How many such factors has 98,010?

(Hint: The integral factors of a number are obtained by multiplying together selections from its prime factors.)

5. How many different parties of 2 or more can be formed from 9 people?

6. Find the number of ways in which at least one book can be selected from 4 identical Spanish books and 5 identical French books.

62. Problems of special types. Some formulas in permutations and combinations are now available. While these are useful, the worst possible use to make of them is to apply them uncritically to any problem that arises. It is necessary to have clearly in mind the precise conditions under which a specified formula gives a correct answer. After a problem has been analysed, it may be found advantageous to make use of formulas, or it may be preferable to reason independently of formulas.

62.1. Mutually exclusive cases. In some problems, we are faced with a number of choices of which any one is admissible, but not more than one simultaneously. Such choices must not be confused with those referred to in the fundamental principle, in which the second operation can always be performed after the first. The choices in question are ***mutually exclusive;*** each one forms the basis of a separate problem, and the final answer is the result of addition rather than of multiplication. This is illustrated in examples 1 and 2.

EXAMPLE 1. In how many ways can a selection be made from 5 books, if at least 3 must be chosen?

Either 3 books or 4 books or 5 books may be chosen so that there are 3 mutually exclusive cases. Three books can be selected in ${}_5C_3$ ways, 4 books in ${}_5C_4$ ways, and 5 books in ${}_5C_5$ ways. Therefore, the total number of selections is

$${}_5C_3+{}_5C_4+{}_5C_5=16.$$

EXAMPLE 2. How many (*a*) combinations,* (*b*) permutations,* 4 at a time, can be formed from the digits 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 5?

Four mutually exclusive cases are provided by the following 4 types of selection:

(1) selections with 3 digits alike and the other different,

(2) selections with 2 digits of one kind and 2 of another,

(3) selections with 2 digits alike and the other 2 different, or

(4) selections with all 4 digits different.

First, we find the number of selections of each type and then determine the number of permutations that each selection will give. The solution is conveniently arranged in tabulated form, thus:

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*The student should make clear to himself why the formulas of sections 60 and 61 are not applicable.

Type	Combinations	Permutations
(1) 3 alike, 1 different	${}_2C_1 \times {}_4C_1 = 8$	$8 \times \frac{4!}{3!} = 32$
(2) 2 alike, 2 alike	${}_4C_2 = 6$	$6 \times \frac{4!}{2!\,2!} = 36$
(3) 2 alike, 2 different	${}_4C_1 \times {}_4C_2 = 24$	$24 \times \frac{4!}{2!} = 288$
(4) all different	${}_5C_4 = 5$	$5 \times 4! = 120$
	Total $= 43$	Total $= 476$

Therefore, 476 different four-digit numbers could be formed from the given 11 integers.

Example 3. In how many ways can 8 identical objects be placed in 3 numbered boxes (*a*) if at least one object must go into each box, (*b*) if one or more of the boxes may be left empty?

(*a*) We think of the 8 identical objects as being arranged in a line, thus:

$$AAAAAAAA.$$

By inserting 2 dividing lines among the objects, we can mark them off into 3 groups, one for each box. For example,

$$AA/AAA/AAA.$$

There are 7 spaces among the 8 objects, and 2 of these must be selected for the purpose of inserting the dividing lines. Therefore, we can mark off 3 groups, each of which contains at least one object, in ${}_7C_2 = 21$ ways.

(*b*) This problem can be solved by observing that the number of ways is now equal to the number of ways of placing $8+3=11$ identical objects in 3 different boxes, so that at least one object goes into each box. For, when the 11 objects are thus divided, we may remove one from each box and thus obtain a division of 8 among the 3 boxes with the possibility that one or two boxes may be left empty; moreover, all possible divisions of the required type are obtainable in this way. Hence, making use of this fact and the method of part (*a*), we find that the number of ways is ${}_{10}C_2 = 45$.

63. The formulas which arise in permutations and combinations provide interesting examples of functions of one or more variables. Some of the properties of these functions are illustrated graphically* in the accompanying figures.

63.1. The factorial function. Consider

$$f(n) = n!\,.$$

This function is defined when n is a positive integer or zero.

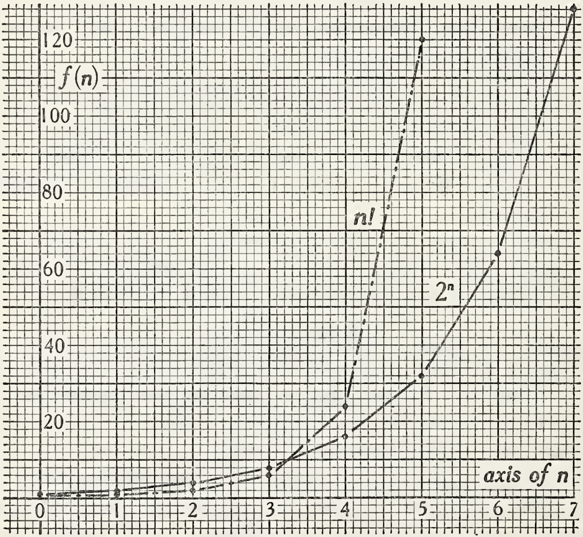

Fig. 25

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*The lines are drawn only to emphasize the relative positions of the points. The functions $n!$ and ${}_nC_r$ have no meaning for other than positive, integral values of n and r.

In fig. 25, the functions $n!$ and 2^n are plotted for positive, integral values of n. For $n=0$, each function takes the value 1 by definition, and for values of n up to 3, $n!$ is less than 2^n. Thereafter, for a given value of n, $n!$ exceeds 2^n and rises very steeply. Although 2^n increases rapidly for increasing values of n, the function $n!$ increases much more rapidly.

63.2. The function ${}_nC_r$. This function depends on the two variables n and r, thus:

$$f(n, r) = {}_nC_r = \frac{n!}{r!(n-r)!}.$$

It is defined for all positive, integral values of n and r, provided $r \leqq n$.

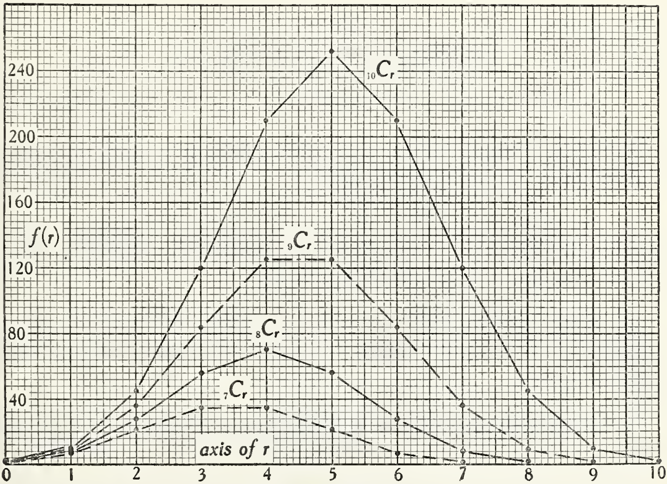

FIG. 26

If n is given a constant value, $f(n, r)$ becomes a function of r; graphs of these functions for $n=7$, 8, 9, and 10 are shown in fig. 26. For a constant value of r, ${}_nC_r$ increases as n increases. This appears, for example, if a vertical line is drawn corresponding to $r=4$. If n is a constant, then as r increases the function increases to a maximum and then decreases, the graph having a vertical axis of symmetry. This is to be expected by virtue of the relation ${}_nC_r = {}_nC_{n-r}$. It is to be observed that, if n is even, the maximum value of ${}_nC_r$ occurs when $r=n/2$; this is illustrated by the graphs of ${}_8C_r$ and ${}_{10}C_r$. On the other hand, if n is odd, there are two central values of r which yield a common maximum of the function. These values are $r=\frac{n-1}{2}$ and $r=\frac{n+1}{2}$, as illustrated by the graphs of ${}_7C_r$ and ${}_9C_r$. These facts can be proved analytically.

EXERCISES ON CHAPTER VII

A

1. In how many ways can three different keys be arranged (*a*) in a row, (*b*) on a key-ring?

2. A committee of 4 is to be selected from 9 available persons. Prove that two specified persons will be included in 1/6 of the total number of possible selections.

3. Find the number of ways in which 8 persons can be assigned to 2 different rooms, if each room must have at least 3 persons in it.

4. Find the number of permutations of the letters of the word *sophomore,* (*a*) taken all together, (*b*) taken 4 at a time.

5. In how many ways can 13 different cards be arranged in a row, if a certain 10 of them must always be in a specified order relative to each other?

6. How many integral factors has the number 1000, excluding unity and the number itself?

7. Find the number of ways in which nine 3's and six 5's can be placed in a row so that no two 5's come together.

8. How many words of 4 letters, beginning with a vowel and ending with a consonant, can be made from the letters of the word *spherical*?

9. Find the number of ways in which a selection of at least one object can be made from 10 different objects.

10. How many changes can be rung with a peal of 8 bells, if a specified bell must be rung first and another specified bell last?

B

11. Find the number of (*a*) combinations, (*b*) permutations of the letters of the word *proportion*, taken 4 at a time.

12. In how many ways can 2 booksellers divide between them 300 copies of one book, 200 copies of another, and 100 copies of a third, if neither bookseller is to get all the books?

13. Find the number of ways in which 12 identical cars can be parked in 4 rows, if each row contains at least one car.

14. How many four-letter words, each beginning with the letter *s*, can be formed from the letters of the word *parallels*?

15. From 10 tennis players, in how many ways can players be selected for a game of singles? In how many ways can teams be selected for a game of doubles?

16. How many numbers with five digits can be formed by using digits from the set 4, 5, 5, 6, 6, 6?

17. Find the number of ways in which 12 people can be seated at two round tables, if 7 are seated at the larger table and 5 at the smaller, and if relative positions only are considered.

18. A company of soldiers consists of a captain, 2 lieutenants, and 20 privates. In how many ways can 5 men be selected so as to include (*a*) at least one officer, (*b*) exactly one officer?

19. In how many ways can 10 similar flags be arranged on a row of 5 flag-poles so that there is at least 1 flag on each pole?

20. In how many ways can 5 red balls, 4 black balls, and 4 white balls be placed in a row so that the balls at each end of the row are of the same colour?

21. Find the total number of automobile licence markers that can be made by using letters and numbers arranged as follows: (*a*) one letter is followed by a number of one, two, three, or four digits, or (*b*) two letters are followed by a number of one, two, or three digits.

22. Find the number of ways in which 12 different cards can be divided evenly among 4 persons. Find also the number of ways in which they can be divided evenly into 4 piles.

23. Find the sum of all three-digit numbers which can be made from the digits 2, 3, 4, 7, 8, 9, repetitions excluded.

24. In how many ways can a committee of 3 be chosen from k married couples, if a husband and his wife cannot both serve on the same committee?

25. A man has 8 different pairs of gloves. In how many ways can he select a right-hand and a left-hand glove that do not match?

26. If 15 points, in general positions in a plane, are joined in all possible ways, find the number of intersections exclusive of the given points.

27. Find the number of (*a*) combinations, (*b*) permutations of the letters of the word *tintinnabulation*, taken 5 at a time.

28. After the manner of fig. 26, represent graphically the function ${}_nP_r$, giving to n the values 2, 3, 4, and 5 in succession and letting r take all admissible values in each case. For what values of n and r is the function ${}_nP_r$ defined? If r is constant, how does the function change with n? If n is constant, what values of r make the function a maximum? If n is constant, is the graph of ${}_nP_r$ symmetrical with respect to any line as in the case of ${}_nC_r$?

C

29. In how many ways can 7 similar algebra books be given to 4 boys, if one or more of the boys may be left without a book?

30. Six chairs bearing numbers from 1 to 6 are provided with cushions whose numbers match those of the chairs on which they rest. In how many ways can these chairs, with cushions, be arranged in a circle (*a*) if the number of a chair and its cushion must always be the same, (*b*) if the odd-numbered cushions must be on the even-numbered chairs?

31. Of 10 oarsmen, 5 can row only on the starboard side of the boat, 4 only on the port side, and 1 on either side. Find the number of ways in which a crew for a four-oared shell can be chosen from them.

32. Find the number of different sums obtainable by making a selection from five banknotes whose denominations are as follows: $1, $2, $5, $10, $20. In how many of these sums would the one dollar bill appear? Find the total value of all the different sums obtainable.

33. From 10 of his friends a man wishes to invite a party to dinner. How many should he decide to invite in order to give himself the greatest possible number of selections? How many selections would he have in this case?

34. The streets of a city consist of m streets running north and south and n streets running east and west. In how many ways can a man walk from the northeast to the southwest corner, if he always walks the least possible number of blocks?

35. Find the number of permutations of the letters of the word (*a*) *intuition*, taken 4 at a time, (*b*) *circumference*, taken 5 at a time.

36. In how many ways can p plus signs and m minus signs be arranged in a row so that no two plus signs are together?

$$(m+1 \geqq p)$$

37. Find in how many of all the possible arrangements of 52 cards in a row, 5 specified cards will be found in the first 13 cards of the row.

38. An automobile parking lot has r rows for parking cars and there are n identical cars to be parked, where $n \geqq r$. If each row can hold n cars, in how many ways can the cars be parked provided that (*a*) at least 1 car is put in each row, (*b*) 1 or more rows may be left empty? In how many ways can n different cars be parked in the r rows, if at least 1 car is put in each row and if the order of the cars in the rows is taken into account?

39. Of the 8! numbers formed by permuting the integers from 1 to 8 inclusive, how many are between

16,500,000 and 32,000,000?

40. (*a*) Prove that ${}_nC_r = \left\{\frac{n+1}{r} - 1\right\} {}_nC_{r-1}$.

(*b*) Use the formula of part (*a*) to show that, for a given value of n, ${}_nC_r$ is a maximum when $r = \frac{n}{2}$, if n is even; and when $r = \frac{n+1}{2}$ or $\frac{n-1}{2}$, if n is odd.

41. A merchant has $4b$ different books to pack in 4 boxes, b books in each box. In how many ways can the packing be done (*a*) if the 4 boxes are numbered differently, (*b*) if the 4 boxes are identical?

42. Find the number of four-digit numbers, each of which contains at least one odd digit and one even digit, repetitions of a digit being permitted.

43. Eight different objects are to be distributed to 3 persons so that each person receives at least one object. In how many ways can this be done?

44. In how many ways can 7 white balls and 6 black balls be placed tightly in a row so that there are just 5 contacts of black with white balls?

CHAPTER VIII

MATHEMATICAL INDUCTION

***64. The purpose of mathematical induction. Illustration of the method.** The subject of this chapter is a useful method of proof known as *mathematical induction.* This method has the disadvantage that it does not provide a way of discovering theorems; its usefulness is in proving a certain class of theorems, the truth of which has been suspected or inferred by other means. Such propositions usually state that a function, $f(n)$, possesses a certain property for positive, integral values of n.

For example, suppose we wish to prove that

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1}=\frac{a(r^n-1)}{r-1}.$$

Denote the right member by $f(n)$. We first note that, if $n=1$,

$$f(1)=\frac{a(r-1)}{r-1}=a.$$

Therefore, $f(n)$ correctly gives the sum of the series when $n=1$. We now assume that $f(n)$ is a valid formula for the sum of the series when n has some fixed value k, so that

$$a+ar+ar^2+\cdots+ar^{k-1}=\frac{a(r^k-1)}{r-1}=f(k).$$

We do not *assume* that the formula holds for any value of n other than $n=k$. Let us see what we can *prove* from our assumption. If we add t_{k+1} or ar^k to each side of the last equation, we get

$$\begin{aligned} a+ar+ar^2+\cdots+ar^{k-1}+ar^k &= \frac{a(r^k-1)}{r-1}+ar^k \\ &= \frac{ar^k-a+ar^{k+1}-ar^k}{r-1} \\ &= \frac{a(r^{k+1}-1)}{r-1} \\ &= f(k+1). \end{aligned}$$

Therefore, if $f(k)$ gives the correct sum for k terms of the series, $f(k+1)$ gives the correct sum for $k+1$ terms. In other words, if the formula, $f(n)$, holds for a certain value of n, it holds also for the next higher value. Now we have shown by trial that $f(n)$ is a valid formula for $n = 1$. Hence, it is valid for $n = 2$. Since it is valid for $n = 2$, it is also valid for $n = 3$, and so on for all positive, integral values of n.

★65. **The three steps of mathematical induction.** The foregoing example illustrates the three steps in a proof by mathematical induction as follows:

(*A*) Show, by actual trial, that $f(n)$ has the required property when n takes the lowest admissible value, which is usually $n=1$.

(*B*) Assume that $f(k)$ has the required property, where k is some fixed integer, and on this assumption prove that $f(k+1)$ also has this property. In other words, if the proposition is true for a certain value of n, say $n=k$, we show that it is also true for the next higher value of n, namely, $n=k+1$.

(*C*) Complete the induction. Since, by part (*A*), the proposition is true for $n=1$, it is also true for $n=2$ by virtue of part (*B*). Since it is true when $n=2$, it is true when $n=3$, and so on for all positive, integral values of n. Therefore, $f(n)$ has the required property for all positive, integral values of n.

It is to be noted that steps (*A*) and (*B*) are *both* necessary for a complete proof. The following observations emphasize this fact.

In natural science and in many life situations, a method of induction is often used by which a general conclusion is drawn from a number of particular cases. Such conclusions, however, are not *necessary* conclusions, and such induction is to be distinguished from mathematical, or complete, induction. The fallacy of drawing general conclusions from particular cases is well illustrated by a property of the expression n^2-n+41. This expression can be shown by actual substitution to yield a prime number for each of the values $n=1, 2, 3, \cdots, 40$. As this property holds in the first forty cases, one is tempted to conclude that it holds in all cases. However, if $n=41$, the expression is equal to 41^2 which is not a prime number.

Likewise step (*B*) alone is insufficient to guarantee a proof. For example, consider the theorem: The sum of the first n even numbers is an odd number. If this theorem is true for $n=k$, then it is true for $n=k+1$, since the addition of an even number to an odd number gives an odd number. However, step (*A*) cannot be demonstrated in this case, and the theorem is false.

EXAMPLE 1. Prove that for all odd, positive, integral values of n, $f(n)=a^n+b^n$ is divisible by $a+b$.

(*A*) When $n=1$, we get $f(1)=a+b$, and the theorem is obviously true.

(*B*) Suppose that $f(k)=a^k+b^k$ is divisible by $a+b$, where k is some odd, positive integer. Then we have

$$f(k+2)=a^{k+2}+b^{k+2}=a^2a^k+b^2b^k$$
$$=a^2a^k+a^2b^k-a^2b^k+b^2b^k=a^2(a^k+b^k)-b^k(a^2-b^2).$$

Now a^2-b^2 is divisible by $a+b$ and, by our assumption, a^k+b^k is also. It follows that $f(k+2)$ is divisible by $a+b$. Therefore, if $f(n)$ is divisible by $a+b$ for any odd, positive, integral value of n, it is divisible when n takes the next higher odd value.

(*C*) In (*A*), the proposition has been found to be true for $f(1)$. Therefore it is true for $f(3)$ and hence true for $f(5)$, and so on for every odd, positive, integral value of n.

Example 2. Prove that the sum of the roots of the equation

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$$

is $-\dfrac{a_{n-1}}{a_n}$.

(*A*) When $n=1$, the equation is $a_1x+a_0=0$, which was the single root $-\dfrac{a_0}{a_1}$. Hence the formula $-\dfrac{a_{n-1}}{a_n}$ gives the sum of the roots when $n=1$.

(*B*) Take any equation of degree $k+1$, which may be written, for brevity, $P(x)=0$. By the fundamental theorem of algebra, this equation has a root. Denote a root by b. Then, by the factor theorem, $P(x)$ contains the factor $x-b$. The other factor is a polynomial of degree k. Hence $P(x)=0$ may be written

$$\text{(i)}\quad (x-b)(a_kx^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0)=0.$$

Now the roots of this equation consist of b and the roots of

$$\text{(ii)}\quad a_kx^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0=0.$$

We now assume that the formula for the sum of the roots holds for any equation of degree k, so that the sum of the roots of (ii) is $-\dfrac{a_{k-1}}{a_k}$. It follows that the sum of the roots of (i) is

$$-\frac{a_{k-1}}{a_k}+b=-\frac{a_{k-1}-ba_k}{a_k}.$$

If the factors of (i) are multiplied out, we get

$$a_k x^{k+1} + (a_{k-1} - b a_k) x^k + \cdots - a_0 b = 0.$$

Hence the sum of the roots of (i) is the result obtained when the formula $-\frac{a_{n-1}}{a_n}$ is applied to this equation. Therefore, if this formula gives the sum of the roots for any equation of degree k, it does also for any equation of degree $k+1$.

(*C*) By (*A*), the formula $-\frac{a_{n-1}}{a_n}$ gives the sum of the roots when $n=1$. Hence, by (*B*), it gives the sum when $n=2$; therefore, it gives the sum when $n=3$, 4, 5, and so on.

EXERCISES ON CHAPTER VIII

(Use mathematical induction for all the proofs in these exercises.)

A

1. Show that the nth odd number is $2n-1$.

2. Prove that $a+(a+d)+(a+2d)+(a+3d)+\cdots$ to n terms
$$= \frac{n}{2}\{2a+(n-1)d\}.$$

3. Prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots$ to n terms
$$= \frac{n}{3}(n+1)(n+2).$$

4. Prove that $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

5. Prove that $1^3+2^3+3^3+\cdots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

6. Prove that $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$.

7. Prove that $2^1\cdot 2^2\cdot 2^3\cdots 2^n=\sqrt{2^{n(n+1)}}$.

B

8. Show that a^m-b^m is divisible by $a-b$ for all positive, integral values of m, and by $a+b$ when m is an even, positive integer.

9. Prove that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$ to n terms $=\frac{n}{n+1}$.

10. Prove that m^2+7m is always divisible by 2, if m is a positive integer.

11. Show that the sum of the interior angles of a polygon of n sides is $2n-4$ right angles.

12. If the product $(a_1+b_1)(a_2+b_2)(a_3+b_3)\cdots(a_n+b_n)$ is multiplied out, show that

(*a*) each term in the result will be of the nth degree in the letters involved, and

(*b*) each term in the result will contain one and only one letter from each of the n factors.

13. Attempt to prove by mathematical induction that $3n^2-6n+4$ is divisible by 3 for positive, integral values of n. What step of the proof can be carried through, and at what point does the proof break down?

14. The following (incorrect) formulas are suggested for the number of right angles in the sum of the angles of a polygon of n sides: (*a*) $2n-3$, (*b*) $3n-7$. Attempt to build up a proof by mathematical induction for each formula. In each case, which step of the proof can be carried through and at which step does the proof break down?

C

15. Prove that $1+(2^2-2)+2+(4^2-4)+3+(6^2-6)+\cdots$ to $2n$ terms is equal to $\dfrac{n(n+1)(8n+1)}{6}$.

16. Prove that for any positive, integral value of n

$$\frac{a^{2n+1}+b^{2n+1}}{a+b}=a^{2n}-a^{2n-1}b+a^{2n-2}b^2-\cdots-ab^{2n-1}+b^{2n}.$$

17. Show that the sum of n terms of the series

$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\cdots \text{ is } \frac{n(n+1)}{2(n^2+n+1)}.$$

18. Prove that

$$\begin{aligned}(a_1+a_2+\cdots+a_n)^2 &= a_1{}^2+a_2{}^2+\cdots+a_n{}^2\\ &\quad+2(a_1a_2+a_1a_3+\cdots+a_{n-1}a_n).\end{aligned}$$

19. Prove, for positive integral values of n greater than 1, that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}<n.$$

20. Prove, for positive integral values of n, that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}>\frac{n}{2}.$$

CHAPTER IX

THE BINOMIAL THEOREM

66. Introduction. The identities

$$(a+x)^2=a^2+2ax+x^2,$$

and

$$(a+x)^3=a^3+3a^2x+3ax^2+x^3,$$

are of frequent use in elementary algebra. The right members are referred to as expansions of the left. Since these expansions are symmetrical in form, the question arises as to the form of expansions of higher powers of the binomial $a+x$. It will be found that such expansions have important uses; the main justification for their study, however, is to satisfy the desire for generalization which is at the root of many of the advances in algebra, as well as in other branches of mathematics.

The expansions of $(a+x)^4$, $(a+x)^5$, etc., may be successively obtained by multiplication. Apart from the fact that this operation becomes increasingly laborious, it does not readily suggest the law by which the coefficients may be found. It is our object to obtain such a law for finding the coefficients in the expansion of $(a+x)^n$, where n is any positive integer. We shall begin by studying the products of binomials, all of which are distinct.

67. Products of distinct binomials. The following expansions of products are easily verified:

$$(a_1+x_1)(a_2+x_2)=a_1a_2+a_1x_2+x_1a_2+x_1x_2.$$

$$(a_1+x_1)(a_2+x_2)(a_3+x_3)=a_1a_2a_3+a_1a_2x_3+a_1x_2a_3+a_1x_2x_3$$
$$+x_1a_2a_3+x_1a_2x_3+x_1x_2a_3+x_1x_2x_3.$$

From these expansions, it is apparent that

(*a*) in each product, the degree of every term, in the letters involved, is equal to the number of factors multiplied;

(*b*) each term in a product is formed by taking exactly one letter from each factor;

(*c*) the expansion consists of the sum of such terms obtained in all possible ways.

These principles are seen to be true for products involving any number of factors (cf. ex. 12, page 247). Thus, the expansion of

$$(1) \qquad (a_1+x_1)(a_2+x_2)(a_3+x_3)\cdots(a_n+x_n)$$

is the sum of all possible nth-degree terms, each of which is formed by choosing one and only one letter from each of the n factors. Since there are two choices of a letter from each factor, the expanded product will contain 2^n different terms.

68. Powers of the binomial $a+x$. The binomial theorem.

Consider now the product

$$(2) \qquad (a_1+x_1)(a_2+x_2)(a_3+x_3)(a_4+x_4),$$

the expansion of which contains 16 terms. If we replace each factor by $a+x$, these 16 terms fall into groups of like terms which are all of degree 4 in a and x. For example, the terms

$$a_1a_2a_3x_4+a_1a_2a_4x_3+a_1a_3a_4x_2+a_2a_3a_4x_1,$$

combine to give the term $4a^3x$ in the expansion of $(a+x)^4$. The terms of this latter expansion, written without their coefficients and in ascending powers of x, are a^4, a^3x, a^2x^2, ax^3, and x^4; in other words, the terms are of the form $a^{4-r}x^r$, where r takes the values 0, 1, 2, 3, 4. There remains the problem of obtaining the coefficients of these terms. For this purpose, we return to the expansion of (2). The terms in the expansion of (2) which combine to give the term containing $a^{4-r}x^r$ are formed by choosing x's from r of the factors and a's from the remaining $4-r$ factors.

The number of these terms is the number of ways of choosing r x's from the 4 different x's of (2). This number is ${}_4C_r$. Hence, the resulting term in the expansion of $(a+x)^4$ is ${}_4C_r a^{4-r}x^r$. Giving to r the values 0, 1, 2, 3, 4, we get the expansion

$$(a+x)^4 = a^4 + {}_4C_1 a^3x + {}_4C_2\, a^2x^2 + {}_4C_3\, ax^3 + x^4.$$

EXERCISES

1. In the product $(a_1+x_1)(a_2+x_2)\cdots(a_9+x_9)$, state (*a*) the number of terms, (*b*) the degree of each term in a and x.

2. In the expansion of $(c+d)^5$, what is the degree of each term in c and d? Write the terms of this expansion, without their coefficients, in ascending powers of d. What is the general form of these partial terms?

3. After like terms are collected, how many terms are there in the expansion of $(a+x)^{20}$? Write the first four terms of this expansion, without their coefficients, in ascending powers of x.

4. Write the terms, without their coefficients, in the expansion of $(a+x)^6$. Write the general form of these terms in a and x.

Use the method of this section to obtain complete expansions for the following:

5. $(a+x)^5$.
6. $(m+n)^4$.
7. $(a+x)^6$.

We shall now extend the argument used for $(a+x)^4$ to $(a+x)^n$, where n is any positive integer.

68.1. Proof of the binomial theorem. The expansion of $(a+x)^n$ will be obtained if we expand the product

$$(1) \qquad (a_1+x_1)(a_2+x_2)(a_3+x_3)\cdots(a_n+x_n),$$

then drop the subscripts from the letters and collect the terms. From the principles for multiplying distinct binomials (see sec. 67), it follows that each term in the expansion of $(a+x)^n$ is of degree n in a and x, and that the terms, apart from their coefficients, are of the form $a^{n-r}x^r$, where r takes the values 0, 1, 2, 3, $\cdots$, n. The coefficient of $a^{n-r}x^r$ can be obtained by considering the expansion of (1). The terms in the expansion of (1) which combine to give the term containing $a^{n-r}x^r$ are formed by choosing x's from r of the factors and a's from the remaining $n-r$ factors in all possible ways. The number of these terms is the number of ways of choosing r x's from the n different x's of (1). This number is ${}_nC_r$. Hence, the resulting term in the expansion of $(a+x)^n$ is

$$(3) \qquad {}_nC_r a^{n-r}x^r.$$

Giving to r the values 0, 1, 2, 3, $\cdots$, n, and adding, we get the expansion:

$$(4) \qquad \boxed{\begin{aligned}(a+x)^n = a^n + {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 + \cdots\\ + {}_nC_r a^{n-r}x^r + \cdots + {}_nC_nx^n.\end{aligned}}$$

69. Proof of the binomial theorem by mathematical induction. Starting with the expansion of equation (4), we may use the method of mathematical induction to prove that this equation is an identity for any positive, integral value of n. Denote the right member of (4) by $F(n)$. We suppose that formula (4) is valid when n takes some positive, integral value k. Therefore,

$$(5) \quad (a+x)^k = a^k + {}_kC_1\, a^{k-1}x + \cdots + {}_kC_r a^{k-r}x^r + \cdots + x^k = F(k).$$

We now prove that in this case $F(n)$ gives the correct expansion when n takes on the next higher value, namely $k+1$. This fact is demonstrated by multiplying both sides of (5) by $a+x$. This gives

$$\begin{aligned}(a+x)^{k+1} &= a^{k+1} + a^kx(1+{}_kC_1) + \cdots + a^{k-r+1}x^r({}_kC_{r-1}+{}_kC_r)\\ &\qquad + \cdots + x^{k+1}\\ &= a^{k+1} + {}_{k+1}C_1a^kx + \cdots + {}_{k+1}C_r\, a^{k+1-r}x^r + \cdots + x^{k+1}\\ &= F(k+1). \qquad \text{(example 2, page 219)}\end{aligned}$$

Hence, if $F(n)$ is a valid expansion for $n=k$, it is valid for $n=k+1$. By trial, $F(n)$ is a correct expansion when $n=1$. Therefore, it is correct when $n=2$. Since it is true for $n=2$, it is true for $n=3$ and so on for any positive integral value of n.

70. Forms of the binomial theorem. The expression $(a+b)^n$, where $a\neq 0$, may be put into the form $a^n\left(1+\frac{b}{a}\right)^n$. It is therefore sufficient to consider the expansion of forms of the type $(1+x)^n$; this results in a gain in simplicity. From (4), we find that

$$\boxed{(1+x)^n=1+{}_nC_1x+{}_nC_2x^2+{}_nC_3x^3+\cdots+{}_nC_rx^r+\cdots+x^n.} \tag{6}$$

Replacing the coefficients by their values in terms of n, we get

$$\boxed{\begin{aligned}(1+x)^n &=1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots\\ &\qquad+\frac{n(n-1)\cdots(n-r+1)}{r!}x^r+\cdots+x^n.\end{aligned}} \tag{7}$$

The following examples illustrate the use of the binomial theorem.

Example 1. Expand $(1-2x)^5$.

From formula (7), we obtain

$$\begin{aligned}(1-2x)^5 &= 1+\frac{5}{1}(-2x)+\frac{5\cdot 4}{1\cdot 2}(-2x)^2+\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(-2x)^3\\ &\quad+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}(-2x)^4+(-2x)^5\\ &=1-10x+40x^2-80x^3+80x^4-32x^5\,.\end{aligned}$$

EXAMPLE 2. Expand $(a-x^3)^{10}$ to 4 terms.

After factoring out a, we use formula (7) as follows:

$$\begin{aligned}(a-x^3)^{10} &= a^{10}\left(1-\frac{x^3}{a}\right)^{10}\\ &= a^{10}\left\{1+\frac{10}{1}\left(\frac{-x^3}{a}\right)+\frac{10\cdot 9}{1\cdot 2}\left(\frac{-x^3}{a}\right)^2\right.\\ &\qquad\left.+\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}\left(\frac{-x^3}{a}\right)^3+\cdots\right\}\\ &= a^{10}\left\{1-10\frac{x^3}{a}+45\frac{x^6}{a^2}-120\frac{x^9}{a^3}+\cdots\right\}\\ &= a^{10}-10a^9x^3+45a^8x^6-120a^7x^9+\cdots .\end{aligned}$$

EXAMPLE 3. Show that $1+{}_nC_1+{}_nC_2+\cdots+{}_nC_n=2^n$.

From formula (6), we have

$$(1+x)^n=1+{}_nC_1x+{}_nC_2x^2+\cdots+{}_nC_nx^n.$$

Since this is true for any value of x, n being a positive integer, we may set $x=1$. Then,

$$(1+1)^n=1+{}_nC_1+{}_nC_2+\cdots+{}_nC_n\,,$$

which proves the proposition.

The numbers $1,\ {}_nC_1,\ {}_nC_2,\ \cdots,\ {}_nC_n$ are called the binomial coefficients and are frequently written

$$c_0,\ c_1,\ c_2,\ c_3,\ \cdots,\ c_n,$$

where no ambiguity can arise.

The student should observe that example 3 merely establishes a fact already proved in chapter VII, namely, that the total number of ways in which a selection can be made from n different things is 2^n-1. (sec. 61.2, page 230.)

EXERCISES

Use the binomial theorem to expand the following:

1. $(1+m)^6$. **2.** $(1+2a)^5$. **3.** $(1+p)^7$.

4. $(1-3x)^4$. **5.** $(1-s^2)^5$. **6.** $(2+4x)^4$.

7. $(m+n)^6$. **8.** $(2+2a)^6$. **9.** $(3-6x)^5$.

10. $(\frac{3}{4}a+1)^4$. **11.** $(3-6x^2)^3$. **12.** $(a+ar)^6$.

13. $(1-a^2)^7$. **14.** $(x^2-x^3)^5$. **15.** $(a-b)^5$.

16. $\left(\sqrt{k}-\frac{k^{\frac{3}{2}}}{m}\right)^6$. **17.** $(z^3-b^2)^5$. **18.** $\left(1+\frac{a^2}{a^{\frac{3}{4}}}\right)^4$.

19. $\left(\frac{x^{\frac{3}{2}}}{\sqrt{x}}-\frac{x^{\frac{3}{4}}}{x^{-\frac{5}{4}}}\right)^6$.

Expand each of the following to 4 terms:

20. $(1-x^2)^{40}$. **21.** $(1/m+m)^{17}$. **22.** $(1+.03)^{10}$.

Expand each of the following to 3 terms:

23. $(1+m\sqrt{2})^{100}$. **24.** $(1/\sqrt{a}-\sqrt{a})^{30}$.

25. $(x^2-y^2)^{12}$. **26.** $(\sqrt{2}+4)^8$.

27. By using the fact that $11^5=(1+10)^5$, compute the value of 11^5. In a similar way, compute the value of (*a*) 99^4, (*b*) 101^4.

28. Using formula (6), prove that

$$c_0+c_2+c_4+\cdots=c_1+c_3+c_5+\cdots.$$

71. The general term. The $(r+1)$th term of a binomial expansion is called the *general term* and may be denoted by t_{r+1}. If x and n are known constants, the general term in the expansion of $(1+x)^n$ becomes a function of r according to the following formula:

$$\boxed{\begin{aligned} t_{r+1} &= {}_nC_r x^r && (8)\\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r\,. && (9)\end{aligned}}$$

It is to be noted that in this expression for the general term, the integer r is *one less than the number of the term.*

Example 1. Find the general term in the expansion of $(a-x^3)^{10}$.

Since $(a-x^3)^{10}=a^{10}\left(1-\dfrac{x^3}{a}\right)^{10}$, formula (8) gives

$$t_{r+1}=a^{10}\left\{{}_{10}C_r\left(\frac{-x^3}{a}\right)^r\right\}=(-1)^r\,{}_{10}C_r\,x^{3r}\,a^{10-r}\,.$$

This expression exhibits the $(r+1)$th term as a function of r which might be denoted by $f(r)$. Then,

$$t_4=f(3)=(-1)^3\,{}_{10}C_3x^9a^7=-120a^7x^9,$$

which checks with example 2, sec. 70.

Example 2. Does x^5 occur in any term in the expansion of $(x^2+1/x^3)^{10}$? Do x^0 or x^4 occur?

We get information concerning particular terms by examining the general term. We have

$$(x^2+1/x^3)^{10}=x^{20}\,(1+1/x^5)^{10}\,.$$

Therefore, $t_{r+1}=x^{20}\{{}_{10}C_r\,(1/x^5)^r\}={}_{10}C_r\,x^{20-5r}$.

In order that the index of the power of x in this term may be 5, $20-5r$ must be equal to 5, whence $r=3$. Thus, x^5

occurs in t_4. Similarly, x^0 occurs in t_5. However, no term in this expansion contains x^4, because $20-5r$ cannot equal 4 unless r is a fraction and such a value is not admissible.

EXERCISES

1. How many terms are there in the expansion of $(1+x)^{25}$? Write down in unsimplified form (*a*) the 5th term and the 24th term in the expansion, (*b*) the coefficient of the 18th term, (*c*) the coefficient of x^8 and of x^{10}.

For each of the following expansions, write t_{50}, t_{83}, the coefficient of t_{21}, and the coefficient of x^{75}.

2. $(1+x)^{1000}$. **3.** $(1-x)^{100}$.

Find the general term in each of the following expansions:

4. $(1+y)^{80}$. **5.** $(1-3x)^{15}$. **6.** $(5-10x)^{20}$.

7. $(a+x^2)^{18}$. **8.** $(a^3+1/a^2)^{10}$. **9.** $(1/n-3n^2)^{24}$.

10. In the expansion of ex. 8, find the term in which a does not occur.

11. Which term in the expansion of ex. 9 contains (*a*) n^{-15}? (*b*) n^{10}?

12. Find t_{r+1} in the expansion of $(x-1/x^2)^{20}$. Use the result to find t_4 and t_{18}. In which term does x^2 occur?

13. Write the 10th term in the expansion of $(1-x^2/2)^{11}$.

14. In the expansion of $(m/n+n/m)^8$, find t_5 and t_7.

15. Find the general term in the expansion of $(1+m\sqrt{2})^{100}$. Check your work by letting $r=2$, and comparing the result with the third term obtained in ex. 23 page 255.

16. Find the general term in the expansion of $(z^2-2z^5)^{10}$. Find also the middle term and the 4th term. What is the numerical coefficient of the 8th term? In which term does z^{26} occur? Does z^{30} occur in this series?

17. Write the following:

(*a*) the 5th term in the expansion of $(1+2c^3)^{10}$;

(*b*) the 18th term in the expansion of $(k/s^2-s/k^2)^{20}$;

(*c*) the 7th term in the expansion of $(\sqrt{2x}+2x^2)^8$.

18. The pth term in the binomial expansion of $(a^2-b)^{21}$ has the indices of a and b equal. Find p and the coefficient of the pth term.

19. Find (*a*) the middle term, (*b*) the 4th term from the end, in the expansion of $(k-1/k^3)^{16}$. Is there a term in this expansion which is independent of k? In which terms do k^{-8} and k^5 occur?

20. Find the general term in the expansion of $(m/n-n^2/m)^6$. In which term does n^6/m^2 occur? Use the expression for the general term to find t_4.

21. In the expansion of $(a-b)^{30}$, find t_{30-r}, t_{2s+1}, t_{x+8}, and t_{n-1}.

22. In the expansion of $(1+x)^n$, show that the coefficient of the pth term from the beginning is equal to the coefficient of the pth term from the end. Under what condition on m and k are the coefficients of the mth and kth terms equal in the expansion of $(1+x)^{60}$? For what values of q is the coefficient of t_{q+1} the same as that of t_8?

23. Show that the coefficient of the 25th term in the expansion of $(1+x)^{50}$ is equal to the sum of the coefficients of the 24th and 25th terms in the expansion of $(1+x)^{49}$. (Refer to page 219, example 2.)

72. Expansion of $(1+x)^n$ for any value of n. In section 70. two forms of the binomial expansion were given as follows:

$$(6)\quad (1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + {}_nC_3x^3 + \cdots + {}_nC_rx^r + \cdots + x^n;$$

$$(7)\quad (1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots + x^n.$$

Each of these equations is an identity in x so long as n is a positive integer, and the proofs which have been given apply only to this case. Suppose now that n is a negative integer or a positive or negative fraction. The right member of (6) is then without meaning since a symbol such as ${}_{-5}C_2$ is not defined. Any term in the right member of (7) will still have a definite value, but the terms of the series will go on indefinitely. This is evident from the nature of the coefficients, which can never contain a zero factor unless n is a positive integer. Therefore, (7) gives an infinite series if n is negative or fractional or both. Two questions now arise:

(*a*) Has this infinite series a sum (in the extended sense)?

(*b*) If so, does this sum give the correct value for $(1+x)^n$?

A complete investigation of this problem is beyond the scope of this book. However, the consideration of a special series will suffice to show that formula (7) does not always provide a valid expansion for $(1+x)^n$ when n is not a positive integer. By setting $n=-1$, we get from (7) the following series which corresponds to $(1-x)^{-1}$:

$$1+\frac{(-1)}{1}(-x)+\frac{(-1)(-2)}{1\cdot 2}(-x)^2+\frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}(-x)^3+\cdots$$

or

$$(10)\qquad 1+x+x^2+x^3+\cdots.$$

Although the function $(1-x)^{-1}$ gives rise to this series, we do not write the series equal to $(1-x)^{-1}$ since, as yet, we have answered neither the question as to whether the series has a meaning nor the question as to whether, if it has a meaning, its value is $(1-x)^{-1}$. For the present example, we are in a position to answer these questions. The series (10) is an infinite geometric series. We have seen (chapter VI, sec. 49.1) that this series converges and has the sum $\dfrac{1}{1-x}$ or $(1-x)^{-1}$ provided that $|x|<1$. For all other values of x, i.e., for $|x|\geqq 1$, the infinite series has no sum and hence cannot represent $(1-x)^{-1}$. For instance, if $x=1$, neither the function $(1-x)^{-1}$ nor the series (10) has any value. If $x=-1$, 2, -3, etc., the function has a value for each value of x, but the series has not.*

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*The student may find his curiosity still unsatisfied regarding the fact that series (10) represents $(1-x)^{-1}$ for $-1<x<1$, and fails completely to do so when x is numerically greater than 1. The following observations may throw light on this question. $(1-x)^{-1}=\dfrac{1}{1-x}$, and if the division indicated in this traction is carried out, we have

$$(1-x)^{-1}=1+x+x^2+\frac{x^3}{1-x}.$$

$$\begin{array}{l} 1-x)\;1\qquad(1+x+x^2 \\ \qquad\;\underline{1-x} \\ \qquad\quad x \\ \qquad\quad\underline{x-x^2} \\ \qquad\qquad x^2 \\ \qquad\qquad\underline{x^2-x^3} \\ \qquad\qquad\quad x^3 \end{array}$$

Similarly,

$$(1-x)^{-1}=1+x+x^2+\cdots+x^{k-1}+\frac{x^k}{1-x}.$$

This equation is true for any positive, integral value of k. As k becomes larger and larger, the quotient part of the right member has more and more terms. The remainder term $\dfrac{x^k}{1-x}$ is, so to speak, pushed off farther and farther. How does the size of the remainder term vary as the series lengthens out? If $|x|<1$, the term $\dfrac{x^k}{1-x}$ dwindles away in magnitude as k increases so as to become as near to 0 as we choose. If, on the other hand, $|x|>1$, the term $\dfrac{x^k}{1-x}$ increases in magnitude as k increases so that the farther the series is continued, the more necessary it is not to lose sight of the remainder term. As an exercise, use (7) to find the infinite series corresponding to $(1-x)^{-2}$. Obtain by division the first few terms and a remainder term. For what values of x does the infinite series represent the function?

The foregoing example makes plausible the truth of the following theorem, which is here stated without proof:

If n is any real number other than 0 or a positive integer, the expansion

$$(11)\quad (1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots,$$

in an infinite series, holds if $|x|<1$, *and fails to hold if* $|x|>1$.

It should be emphasized that the expansion (6) has no meaning in this case.

In order to stress further the distinction between the two cases of the binomial theorem which arise according as n is or is not a positive integer, the salient facts are recorded in the following table.

If n is a positive integer or 0	If n is not a positive integer or 0
The expansion of $(1+x)^n$ contains $n+1$ terms.	The series (11) to which $(1+x)^n$ gives rise is an infinite series.
The expansion has the same value as $(1+x)^n$ for all values of x.	The infinite series (11) represents $(1+x)^n$ if $|x|<1$, and fails to do so if $|x|>1$.*

73. Examples. The following problems illustrate the use of the binomial theorem for other than positive, integral values of n.

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*If $x=1$ or -1, the expansion does or does not hold, depending on the value of n. The investigation of this question belongs to more advanced mathematics.

Example 1. Write down the first 3 terms and the general term in the expansion of

$$\frac{1}{\sqrt[3]{8-x^2}},$$

in ascending powers of x. For what values of x does the expansion represent the function?

We have

$$\frac{1}{\sqrt[3]{8-x^2}}=(8-x^2)^{-\frac{1}{3}}=\frac{1}{2}\left(1-\frac{x^2}{8}\right)^{-\frac{1}{3}}$$

$$=\frac{1}{2}\left\{1+\frac{(-\frac{1}{3})}{1}\left(\frac{-x^2}{8}\right)+\frac{(-\frac{1}{3})(-\frac{4}{3})}{1\cdot2}\left(\frac{-x^2}{8}\right)^2+\cdots\right\}$$

$$=\frac{1}{2}\left\{1+\frac{x^2}{24}+\frac{x^4}{288}+\cdots\right\}$$

$$=\frac{1}{2}+\frac{x^2}{48}+\frac{x^4}{576}+\cdots.$$

This series will represent the given function provided

$$\left|\frac{-x^2}{8}\right|<1, \text{ or } x^2<8, \text{ or } -\sqrt{8}<x<\sqrt{8}.$$

Also, the general term is

$$\frac{1}{2}\left\{\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})\cdots(-\frac{1}{3}-r+1)}{r!}\left(\frac{-x^2}{8}\right)^r\right\}$$

$$=\frac{1}{2}\left\{(-1)^{2r}\frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\frac{3r-2}{3}}{r!}\frac{x^{2r}}{8^r}\right\}$$

$$=\frac{1\cdot4\cdot7\cdots(3r-2)}{3^r\cdot2^{3r+1}\cdot r!}x^{2r}.$$

Check:

$$t_3=\frac{1\cdot4}{3^2\cdot2^7\cdot2!}x^4=\frac{x^4}{576}, \text{ as before.}$$

EXAMPLE 2. Find an expression for the general term in the expansion of $(1-x)^{-n}$, where n is a positive integer.

Using formula (9), we have

$$t_{r+1}=\frac{-n(-n-1)(-n-2)\cdots(-n-r+1)}{r!}(-x)^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r$$

$$=\frac{(n+r-1)!}{(n-1)!r!}x^r.$$

If x is a constant, this formula gives the general term as a function of n and r which might be denoted by $F(n, r)$. Expansions of this type, in which the index is a negative integer, are common in practice. Their general terms can be obtained independently or by using $F(n, r)$. Thus, in the expansion of $(1-x)^{-3}$, we have

$$t_{r+1}=F(3, r)=\frac{(r+2)!}{2!\,r!}x^r=\frac{(r+1)(r+2)}{2}x^r.$$

If we assign to r in succession the values 0, 1, 2, 3, . . . , we obtain the expansion (14) below. Similarly, expansions of $(1-x)^{-1}$ and $(1-x)^{-2}$ in ascending powers of x are obtained. On account of their interest, these three expansions are here given:

(12) $(1-x)^{-1}=1+x+x^2+x^3+\cdots+x^r+\cdots$;

(13) $(1-x)^{-2}=1+2x+3x^2+4x^3+\cdots+(r+1)x^r+\cdots$;

(14) $(1-x)^{-3}=1+3x+6x\ +10x\ +\cdots$

$$+\frac{(r+1)(r+2)}{2}x^r+\cdots.$$

The student is advised to make himself familiar with these expansions. It is to be noted that the equality sign holds in each case only for the range of values $-1<x<1$.

EXAMPLE 3. Find the first 3 terms and obtain the term containing x^r as a function of r in the expansion, in ascending powers of x, of

$$\frac{1+3x}{(1+x)^2}.$$

We have

$$\begin{aligned}\frac{1+3x}{(1+x)^2} &= (1+3x)(1+x)^{-2}\\ &= (1+3x)(1-2x+3x^2-\cdots+(r+1)(-x)^r+\cdots)\\ &= (1-2x+3x^2-\cdots)+(3x-6x^2+\cdots)\\ &= 1+x-3x^2+\cdots .\text{*}\end{aligned}$$

The first 3 terms are obtained by multiplying out the partial product as far as the terms involving x^2. Terms with x^r will occur when 1 and $3x$ are multiplied by the terms with x^r and x^{r-1}, respectively, in the expansion of $(1+x)^{-2}$. Therefore, the term containing x^r is

$$\begin{aligned}1\,(r+1)(-x)^r+3x\cdot r\,(-x)^{r-1} &= (-x)^r(r+1-3r)\\ &= (-1)^r(1-2r)x^r.\end{aligned}$$

Check: The term containing x^2 is $(-1)^2(1-4)x^2=-3x^2$, as before.

EXAMPLE 4. Find the first 3 terms and t_{r+1} in the expansion of

$$\frac{5x-7}{(x+1)(x-3)}$$

(*a*) in ascending powers of x, (*b*) in descending powers of x. State the condition under which each expansion represents its function.

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*It is here assumed that the product of an infinite series and a polynomial is obtained by the process of term by term multiplication.

(*a*) Separating the fraction into partial fractions as in sec. 37, we obtain

$$\frac{5x-7}{(x+1)(x-3)}=\frac{3}{x+1}+\frac{2}{x-3}=\frac{3}{1+x}+\frac{-\frac{2}{3}}{1-\frac{x}{3}}$$

$$=3\,(1+x)^{-1}-\frac{2}{3}\left(1-\frac{x}{3}\right)^{-1}.$$

Expanding, we have

$$3\{1-x+x^2-\cdots+(-x)^r+\cdots\}$$

$$-\frac{2}{3}\left\{1+\frac{x}{3}+\frac{x^2}{9}+\cdots+\frac{x^r}{3^r}+\cdots\right\}$$

$$=\frac{7}{3}-\frac{29}{9}x+\frac{79}{27}x^2+\cdots+\left\{3\,(-1)^r-\frac{2}{3^{r+1}}\right\}x^r+\cdots. \qquad \text{(i)}$$

The expansion of $(1+x)^{-1}$ is valid for $|x|<1$, and that of $\left(1-\frac{x}{3}\right)^{-1}$ is valid for $\left|\frac{x}{3}\right|<1$, that is for $|x|<3$. Since each of these expansions must be valid in order that the series (i) represent the function, the required condition is $|x|<1$.

(*b*) In order to find an expansion in descending powers of x, we proceed as follows:

$$\frac{5x-7}{(x+1)(x-3)}=\frac{3}{x+1}+\frac{2}{x-3}=\frac{\frac{3}{x}}{1+\frac{1}{x}}+\frac{\frac{2}{x}}{1-\frac{3}{x}}$$

$$=\frac{3}{x}\left(1+\frac{1}{x}\right)^{-1}+\frac{2}{x}\left(1-\frac{3}{x}\right)^{-1}.$$

Expanding, we have

$$\frac{3}{x}\left\{1-\frac{1}{x}+\frac{1}{x^2}-\cdots+\left(\frac{-1}{x}\right)^r+\cdots\right\}$$
$$+\frac{2}{x}\left\{1+\frac{3}{x}+\frac{9}{x^2}+\cdots+\frac{3^r}{x^r}+\cdots\right\}$$
$$=\frac{5}{x}+\frac{3}{x^2}+\frac{21}{x^3}+\cdots+\frac{3(-1)^r+2\cdot 3^r}{x^{r+1}}+\cdots$$
$$=5x^{-1}+3x^{-2}+21x^{-3}+\cdots+\{3(-1)^r+2\cdot 3^r\}x^{-r-1}+\cdots. \quad \text{(ii)}$$

The series (ii) represents the function for values of x which make the expansion of $\left(1+\frac{1}{x}\right)^{-1}$ and of $\left(1-\frac{3}{x}\right)^{-1}$ both true. The former expansion is true when $\left|\frac{1}{x}\right|<1$, that is when $|x|>1$; similarly, the latter expansion is true when $|x|>3$. Hence, if $|x|>3$, both expansions are valid and the series (ii) represents the function.

EXERCISES

A

Write down and simplify the first three terms in the expansion, in ascending powers, of each of the following. State the condition under which each expansion represents its function.

1. $(1+x)^{\frac{3}{4}}$.
2. $(1-x)^{-\frac{1}{2}}$.
3. $(1+2a)^{-4}$.
4. $(1-3p)^{\frac{1}{2}}$.
5. $\sqrt{4+8x}$.
6. $(9-18a)^{\frac{3}{2}}$.
7. $(2-4x)^{-1}$.
8. $(8-2x)^{2/3}$.
9. $(1-x^2)^{-2}$.
10. $\dfrac{1}{(2-3k)^2}$.
11. $\dfrac{1}{\sqrt[3]{27+9a}}$.
12. $\dfrac{1}{(1+x^2)^5}$.

13-24. Find in simplified form the general term in each of the expansions of the preceding 12 exercises. In each case, check your work by setting $r=2$ and comparing the result with t_3 previously obtained.

25. Write down and simplify the first 4 terms in the expansion, in ascending powers of x, of

$$\frac{1}{\sqrt[3]{8+24x^2}}.$$

Under what condition does the expansion represent the function?

26. Find in simplified form the general term of the expansion, in ascending powers of a, of $(27-81a)^{-2/3}$. Use the general term to obtain the first three terms in the expansion.

B

27. Use formula (11) to obtain the first 4 terms and the general term in the expansion of $(1-x)^{-2}$. What is the general term in the expansion of $(1+x)^{-2}$? What information does this latter general term give about the signs of the successive terms in the expansion of $(1+x)^{-2}$? For what values of x do these expansions represent their functions?

28. Use formula (9) to find the general term in the expansion of $(1-x)^{-3}$, and deduce the first 5 terms in the expansion in ascending powers of x. What are the 20th and 62nd terms in this expansion? Find the terms in which x^{17}, x^{100}, x^{n-1}, and x^{3m+1} occur.

29. Expand the function $(1+x)/(1-x)$ to 4 terms in ascending powers of x. Under what condition does the expansion represent the function? Find the term in the expansion containing x^r and use it to check t_4.

Find the first 3 terms in the expansions of the following functions in ascending powers of x.

30. $\dfrac{x-3x^2}{(1-x)^2}$. 31. $\dfrac{x-3x^2}{(1+x)^2}$.

32. $\dfrac{4+5x^3}{(1-x)^3}$. 33. $\dfrac{4+5x^3}{(1+x)^3}$.

34. Find the terms containing x^{100} and x^n in the expansion of ex. 30.

35. Find the terms containing x^{50} and x^r in the expansion of ex. 33.

36. Find the first 4 terms in the expansion of

$$(1+x+x^2)(1-x^2)^3$$

in ascending powers of x. For what values of x is the complete expansion equal to the function?

Find the terms containing x^{50} and x^n in the expansion of each of the following functions.

37. $\dfrac{3-4x+5x^2}{(1-x)^3}$. 38. $\dfrac{x^2+5x^3}{x-x^2}$.

39. Compute $\sqrt{1+.02}$ to 5 figures. Similarly, compute the value of $\dfrac{1}{(1.03)^3}$ to 4 figures.

40. Expand the function $(1-3x)^{\frac{1}{3}}+\dfrac{16}{\sqrt{1+x}}$ in ascending powers of x to 4 terms. For what values of x does the expansion represent the function?

41. Find the first 3 terms and the term containing x^n in the expansion in ascending powers of x of each of the following functions:

$$(a)\ \frac{3}{1-x-2x^2}; \qquad (b)\ \frac{2+13x}{2+5x-3x^2}.$$

State the condition under which each expansion represents its function.

42. Find the general term and deduce the first 4 terms in the expansion of $(1-x)^{-4}$, in ascending powers of x.

43. Show that x^m will occur in the expansion of $\left(x+\frac{1}{x^3}\right)^p$ if $p-m$ is a multiple of 4. Under what condition will there be a term independent of x?

44. Use the binomial theorem to expand $(1-6k+9k^2)^3$ and $(1+p+p^2)^3$.

45. Find the coefficient of x^{100} and of x^8 in the expansion of

$$\frac{2+3x+4x^2}{x^2+x^3}.$$

46. Expand $(x-1)^{-1}$ in descending powers of x to 4 terms. Under what condition does the expansion represent the function which gives rise to it?

74. Use of the binomial theorem in approximations. The binomial theorem provides a useful method of finding approximate values for certain numerical expressions.

Example. Show that $\frac{1}{\sqrt[3]{127}}=.19894$, correct to 5 figures.

We have

$$\frac{1}{\sqrt[3]{127}}=(127)^{-\frac{1}{3}}=(125+2)^{-\frac{1}{3}}=\tfrac{1}{5}(1+.016)^{-\frac{1}{3}}$$

$$=\tfrac{1}{5}\Big\{1-\tfrac{1}{3}(.016)+\frac{(-\frac{1}{3})(-\frac{4}{3})}{1\cdot 2}(.016)^2$$

$$+\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{1\cdot 2\cdot 3}(.016)^3+\cdots\Big\}$$

$$=\tfrac{1}{5}(1-.005333+.000057-.0000007+\cdots)$$

$$=.19894, \text{ approximately.}$$

It should be noted in this solution that

(1) the series expansion is valid because $.016<1$, and

(2) since the 4th term begins with 6 zeros after the decimal point, the first 3 terms give the answer correct to 5 figures.

74.1. If x is so small that x^2 and higher powers are insignificant, then $1+nx$ is a good approximation for $(1+x)^n$. By using this fact, we can replace an expression involving powers of binomials in x by a simple expression which gives a good approximation to it. For example, if x^2 and higher powers are insignificant, we have

$$\frac{(1-4x)^{\frac{1}{2}}+(1+x)^{\frac{2}{3}}}{(4+5x)^{\frac{3}{2}}}=\frac{(1-2x)+\left(1+\frac{2}{3}x\right)}{8\left(1+\frac{5x}{4}\right)^{\frac{3}{2}}}, \text{ approximately,}$$

$$=\frac{2-\frac{4x}{3}}{8\left(1+\frac{15x}{8}\right)}, \text{ approximately,}$$

$$=\left(\frac{1}{4}-\frac{x}{6}\right)\left(1+\frac{15x}{8}\right)^{-1}$$

$$=\left(\frac{1}{4}-\frac{x}{6}\right)\left(1-\frac{15x}{8}\right), \text{ approximately,}$$

$$=\frac{1}{4}-\frac{61}{96}x, \text{ approximately.}$$

EXERCISES

1. Use the binomial theorem to evaluate the following:

(*a*) $\sqrt{26}$ correct to 5 figures; (*b*) $\sqrt[3]{999}$ correct to 6 figures;

(*c*) $\sqrt[6]{64.64}$ correct to 5 figures; (*d*) $(600)^{\frac{3}{4}}$ correct to 4 figures;

(*e*) $\dfrac{1}{\sqrt[5]{3120}}$ correct to 5 figures; (*f*) $\sqrt[3]{(346.43)^2}$ correct to 6 figures.

2. If x^2 and higher powers of x are insignificant, find approximate values of

$$(a)\ (8-x)^{\frac{2}{3}}\cdot(1+4x)^{-\frac{3}{4}};\quad (b)\ \frac{(4+3x)^{\frac{1}{2}}+(16-8x)^{\frac{3}{4}}}{(8+24x)^{\frac{1}{3}}-(1-2x)^{\frac{3}{2}}}.$$

3. An engineer surveys a rectangular field, using a chain that is incorrect by one-tenth of one per cent of its length. Show that the error thus arising in the determination of the area of the field will be approximately two-tenths of one per cent of the area.

EXERCISES ON CHAPTER IX

A

Use the binomial theorem to expand the following. State the condition under which each expansion represents its function.

1. $(1+x)^{\frac{1}{2}}$, to 4 terms. 2. $(x-2a)^5$.

3. $(1+x^3)^{-1}$, to 5 terms. 4. $(1-4x)^{\frac{5}{4}}$, to 4 terms.

5. $\sqrt[3]{8+16x}$, to 4 terms. 6. $(x-3x^2)^{20}$, to 4 terms.

7. $(1/2+x/4)^{-3}$, to 5 terms. 8. $(4-6x)^{-\frac{3}{2}}$, to 3 terms.

Write down and simplify the general term and the fifth term in the expansion of each of the following.

9. $(3-2q)^7$. **10.** $(1+3x)^{-4}$. **11.** $(8-24x)^{\frac{2}{3}}$.

12. What is the coefficient of x^5 in the expansion of $(1-x)^{-\frac{3}{2}}$?

13. Find $\sqrt[3]{128}$ correct to 6 figures.

14. (*a*) Show that, if h is small, the following equation is approximately correct: $(1+h)^m=1+hm$.

(*b*) If h and k are numerically small, prove that

$$\frac{1+h}{1+k}=1+h-k, \text{ approximately.}$$

15. If x is so small that its square and higher powers are insignificant, find linear functions which closely approximate

(*a*) $(1-3x)^{-\frac{2}{3}}$; (*b*) $(4+8x)^{\frac{5}{2}}$.

B

16. Which term in the expansion of $(x^5-1/x^3)^{10}$ contains x^{10}? Is there a term independent of x in this expansion?

17. In the expansion of $(1+x)^{80}$, the coefficients of the $(2p+5)$th and the $(p+8)$th terms are equal. Find the value of p.

18. Find the first 3 terms and the general term in the expansion of

$$\frac{x+2x^2+3x^3}{(1-x)^2},$$

in ascending powers of x. Under what conditions does the expansion* represent the function?

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*In such expansions, the general term may fail to give the first term or the first two terms. This is illustrated by the following example:

$$\frac{a+bx+cx^2}{1-x}=(a+bx+cx^2)(1+x+x^2+\cdots)=a+(a+b)x+(a+b+c)x^2+\cdots$$
$$+(a+b+c)x^r+\cdots.$$

The general term in this expansion is $(a+b+c)x^r$, in the sense that all terms after the second conform to this type.

19. Find the terms containing x^{50} and x^n in the expansion of

$$\frac{x-2x^3}{(1+x)^3}.$$

20. If x^2 and higher powers of x are insignificant, find a linear function which closely approximates

$$\frac{(1-2x)^{2/3}\,(4+5x)^{3/2}}{(1-3x)^{3/4}}.$$

21. In which terms of the expansion of $(s^3+1/s)^{40}$ do the following occur: (*a*) s^0, (*b*) s^{40}, (*c*) s^{25}?

22. Find the qth term from the end in the expansion of

$$(s+1/s)^{20}.$$

23. The first term of a geometric series is 100 and the 101st term is 101. Use the binomial theorem to find the common ratio, correct to 8 figures.

24. Expand $\left(3\sqrt{e}-\frac{e^3}{3}\right)^6$ to 4 terms. Does e^{13} occur in this expansion? Is there a term in the expansion independent of e? For what values of e is the expansion equal to the function?

25. Separate the function $\frac{2x}{1-4x+3x^2}$ into partial fractions and thus find the first 4 terms of its expansion in ascending powers of x. What is the coefficient of x^m in this expansion? For what values of x does the expansion represent the function?

26. The volume of a sphere is 264 cubic inches. Find its radius correct to 3 figures. $(\pi=22/7)$.

27. Find the term in the expansion of $(x^2+k^3)^9$ for which the sum of the indices of x and k is 21.

28. Obtain the first 3 terms and the coefficient of z^{2k} and of z^{2k-1} in the expansion of $\frac{4z-z^3}{4-z^2}$, in ascending powers of z.

29. It is found that when a certain metal is heated until its temperature is increased by 1 °C., the length of a bar of the metal increases by .0000082 of itself. The temperature of a cube of this metal is increased by 1 °C. If each edge of the cube originally measured 1 foot, find, in cubic feet, the approximate increase in the volume of the cube.

30. Find the sum of the coefficients in the expansion of

$$(a)\ (1-3x)^4, \qquad (b)\ (1+2x+3x^2)^5.$$

31. Find a polynomial approximation for $\dfrac{1}{(1-2x)^{1/3}}$ if x^3 and higher powers of x are insignificant. Use this polynomial to evaluate the function when $x=.01$.

32. Find the value of 1.02^{-5} and of $\sqrt[4]{80}$, correct to 4 figures.

33. If x^3 and higher powers of x are insignificant, show that the following equation is approximately correct:

$$\frac{5}{1+x-6x^2}=5\,(1-x+7x^2).$$

34. Find the coefficient of x^{10} and of x^m in the expansion of

$$(3-4x+5x^2)\,(1-x)^{-4}.$$

35. A function of 2 variables is given by $\dfrac{(1-k)^{-\frac{1}{2}}+(1-e)^{-\frac{1}{2}}}{1+\sqrt{(1-k)\,(1-e)}}$. Show that if k and e are so small that k^2, e^2, and ek are insignificant, the expression $\dfrac{k+e+2}{2}$ furnishes a good approximation for the function.

36. Show that $(1-x+x^2)^{-1}=\dfrac{1+x}{1+x^3}$. Make use of this fact to prove that the coefficient of x^8 in the expansion of $(1-x+x^2)^{-1}$ is -1.

37. Find the coefficient of a^5 in the expansion of $(1-3a)^{10/3}$, and the term independent of p in the expansion of $\left(p^2-\frac{1}{p}\right)^6$.

38. Find and simplify the general term in the expansion of $(1-2x^2)^{-1/2}$. Use this information to obtain the first 4 terms of the expansion in ascending powers of x. For what range of values of x is this expansion equal to the function?

39. Show that the middle term in the expansion of $\left(1+\frac{x}{2}\right)^{30}$ is

$$\frac{1\cdot 3\cdot 5\cdots 29}{15!}x^{15}.$$

40. Prove that $\left\{\frac{2576}{2575}\right\}^5$ is approximately equal to $\frac{516}{515}$.

41. The volume of a rectangular tank is computed by measuring its inside length, width, and depth. If these measurements are made with a metre stick that is incorrect by 1/1000 of its length, show that the resulting error in the determination of the volume is approximately 3/1000 of the volume.

42. Expand the following functions to 4 terms in descending powers of x:

(*a*) $(x-1)^{-2}$;   (*b*) $(4+3x)^{-3}$.

For what values of x is each expansion equal to its function?

43. If the function $\frac{(1+3x)^{2/3}}{(1-4x)^{3/4}}$ is expanded in ascending powers of x, find the first 3 terms. For what range of values of x is the expansion equal to the function?

C

44. What is the greatest coefficient in the expansion of

$$(1+x)^{50}?$$

45. Given $p=x+x^{-1}$, show that $p^{-1}=x^{-1}-x^{-3}+x^{-5}-\cdots$ provided $|x|>1$, and that $p^{-1}=x-x^3+x^5-\cdots$ provided $|x|<1$.

46. The first 3 terms of a binomial expansion are

$$1+16x+112x^2.$$

Find the function which gives the expansion.

47. Show that, for a properly chosen range of values of x,

$$\frac{256-8x^2}{(16-x^2)^2}=1+\frac{3x^2}{2^5}+\cdots+\frac{n+2}{2^{4n+1}}x^{2n}+\cdots.$$

Find the range of values.

48. Assuming that the infinite series

$$1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\cdots$$

and

$$1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots$$

can be multiplied as if they were polynomials, show that the coefficient of x^r in the product is

$$\frac{(m+n)(m+n-1)\cdots(m+n-r+1)}{r!}.$$

49. If n is a positive integer, find the coefficient of x^n in the expansion of $\dfrac{(1+x)^n}{1-x}$.

50. Put the binomial series $1+\dfrac{2}{6}+\dfrac{2\cdot 5}{6\cdot 12}+\dfrac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\cdots$ into the form (11) and hence deduce the fact that this series is equal to $\sqrt[3]{4}$.

51. Given that n is a positive integer and that the coefficients of the pth and $(p+1)$th terms in the expansion of $(1+x)^n$ are equal, prove that the coefficients of the $(p-1)$th and the $(p+2)$th

52. Find the sum of the infinite binomial series

$$1+\frac{1}{3^2}+\frac{1\cdot 4}{1\cdot 2}\cdot\frac{1}{3^4}+\frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{1}{3^6}+\cdots.$$

53. By making use of the fact that

$$(1+x)^3\cdot(1+x)^n=(1+x)^{n+3},$$

prove that

$${}_nC_r+3\,{}_nC_{r-1}+3\,{}_nC_{r-2}+{}_nC_{r-3}={}_{n+3}C_r.$$

54. (*a*) In the expansion of $(1+x)^n$, where x and n are positive, show that

$$t_{r+1}=t_r\cdot\left\{\frac{n+1}{r}-1\right\}x.$$

(*b*) Use the result of (*a*) to prove that the terms of the expansion will continue to increase so long as

$$r<\frac{(n+1)x}{1+x}.$$

(*c*) Hence, find the numerically greatest term in the expansion when $n=10$ and $x=1/3$; also, when $n=9$ and $x=1$.

55. (*a*) If n is positive, show that the terms in the expansion of $(1+x)^{-n}$ will continue to increase numerically so long as

$$r<\frac{(n-1)|x|}{1-|x|}.$$

(*b*) Which term in the expansion of $(1-x)^{-10}$ is numerically greatest (i) when $x=2/3$, (ii) when $x=1/3$?

56. Find the value of the numerically greatest term in the expansion of $(a+2x)^n$, if $a=4$, $x=1$, and $n=11/2$.

57. Which term in the expansion of $(1+3x)^{12}$ has the numerically greatest coefficient?

58. Prove the following:

(*a*) $c_1+2c_2+3c_3+\cdots+nc_n=n\,2^{n-1}$;

(*b*) $c_0^2+c_1^2+c_2^2+\cdots+c_n^2=\dfrac{(2n)!}{n!\,n!}$.

59. Prove that the coefficient of z^r in the expansion of

$$\frac{z}{(a-z)(b-z)}$$

in ascending powers of z is

$$\frac{b^{-r}-a^{-r}}{a-b}.$$

60. Find the coefficient of x^{10} and of x^n in the expansion in ascending powers of x of

$$\frac{(1+x^3)(1-x+x^2)}{(1-x^2)(1-x)}.$$

State the range of values of x for which the expansion is equal to the function.

61. Prove that the sum of the first p coefficients in the expansion of $(1-x)^{\frac{n}{k}}$ is equal to the pth coefficient in the expansion of $(1-x)^{\frac{n-k}{k}}$.

62. Find the coefficient of x^{2n+1} in the expansion of $\dfrac{1}{1+3x-4x^3}$ in a series of ascending powers of x.

63. Find the coefficient of x^m in the expansion of

$$(1-3x+6x^2-10x^3+\cdots)^{-m}.$$

CHAPTER X

MATHEMATICS OF INVESTMENT

75. Introduction. Many problems that arise in the investment and use of money are essentially mathematical in character. In the present chapter, we shall be interested in some of these problems which provide an important application of the geometric series. We begin by examining some fundamental ideas.

76. Compound interest. It is generally known that in return for \$1 deposited today, a bank will pay more than \$1 at some future date. The excess paid by the bank at the future date is called the ***interest.*** Thus interest is a rental paid for the use of money. Payments of interest are made not only by banks, but also by governments, corporations, and individuals who borrow money. Interest is calculated as a certain percentage of the sum loaned, or ***principal.*** Thus a rate of 5% per annum means that the interest on \$1 for one year is \$0.05.

According to the method by which it is computed, interest is classed as simple or compound. In simple interest, the interest payment, when due, is added to the debt but the lender continues to receive interest only on his original loan. For example, if \$100 is loaned at 5% per annum, the interest at the end of one year is $\frac{5}{100}$ of \$100 or \$5, and the interest payable after two years is two times $\frac{5}{100}$ of \$100 or \$10. However, it is customary in business practice to add interest, when due, to the principal so that interest is earned on the previously accumulated interest as well as on the principal. Interest computed in this way is called ***compound interest.*** If \$100 is loaned at compound interest of 5% per annum, compounded annually, the \$5 interest due at the end of the first year is added to the principal so that the principal during the second year is \$105. The interest due at the end of the second year is $\frac{5}{100}$ of \$105, or \$5.25; and the total interest for two years is \$10.25. Hence the principal for the third year is \$110.25, and so on. Hereafter, unless the contrary is stated, it may be assumed that interest means compound interest. In the above example, the interest is converted into principal, or compounded, annually. In the same way, whenever there is more than one interest payment date in the duration of a loan, the interest payable is added to the principal or compounded. The time between two consecutive payments of interest is called the ***interest period,*** or more simply, the ***period.*** The interest is compounded annually, semiannually, or quarterly, according as the period is a year, six months, or three months, respectively. In the work that follows, a rate of $n\%$ without further qualification will mean ***n% per annum, compounded annually.***

Since interest calculations are independent of the currency used, we shall investigate the interest on 1, without specifying dollars, pounds, francs, or other currency. The interest on a sum P for a given term and rate is P times the interest on 1; hence, the amount of P is P times that of 1. Let i be the interest on 1 for one period. For example, if the rate is 4% per annum, compounded annually, then $i=.04$ and the period is one year; also, if the rate is 4% per annum, compounded quarterly, $i=.01$ and the period is three months.

A sum of 1 invested at interest i for one period will amount to $1+i$. During the second period, we have principal $1+i$ invested at i; since the sum P invested for one period amounts to $P(1+i)$, therefore the amount of 1 at the end of two periods is

$(1+i)(1+i)$ or $(1+i)^2$. Similarly, during the third period, the principal is $(1+i)^2$ so that the original 1 amounts to $(1+i)^2(1+i)=(1+i)^3$ at the end of three periods. Proceeding in this way, we find that 1 amounts to $(1+i)^n$ at the end of n periods. Therefore, P amounts to $P(1+i)^n$ in n periods. The final sum accumulated from the original principal P is called the ***amount*** of P at interest i for n periods. If we denote this amount by A, we have

$$\boxed{A = P(1+i)^n.} \tag{1}$$

On pages 334-5, the function $(1+i)^n$ is tabulated for values of n from 1 to 40 and for values of i ranging from .005 to .07, that is, from a rate of ½% to a rate of 7%.

EXAMPLE 1. What is the amount of \$550, invested for 10 years at 3%, compounded annually? How much interest is received?

Since $P=\$550$, $i=.03$, and $n=10$, formula (1) gives

$$A = \$550\,(1.03)^{10} = \$550\,(1.343916) = \$739.15, \text{ approx.}$$

Therefore, the interest received is $\$739.15-\$550 = \$189.15$. It should be noted that the interest was determined by first finding the amount and then subtracting the principal from the amount. The interest could be determined without finding the amount from the fact that the interest is

$$A-P=\$550(1.03^{10}-1)=\$500(0.343916)=\$189.15.$$

EXAMPLE 2. If a man deposits £125 at 4% per annum, compounded semi-annually, what amount should be to his credit at the end of 5 years?

The number of half-year periods in 5 years is 10; also, since the rate is 2% per half year, $i=.02$. Hence, we have $P=£125$, $n=10$, and $i=.02$. Therefore, from (1),

$$A = £125\,(1.02)^{10} = £125\,(1.218994) = £152.37,$$

approximately, or £152 7s 5d, approximately.

Example 3. In what time will money double itself at 8% per annum, compounded quarterly?*

Since the given rate is equivalent to 2% per period of 3 months, the problem is to determine in how many periods 1 amounts to 2, if $i = .02$. From the tables, we observe that 1 becomes 1.99989 in 35 periods, and 1 becomes 2.03989 in 36 periods. Interpolating, we find that 1 becomes 2 in

$$35 + \frac{.00011}{.04000} = 35.003$$

periods, or about $8\frac{3}{4}$ years, since a period, in this case, is 3 months.

77. Present Value. Formula (1) may be written in the alternative form

$$\boxed{P = \frac{A}{(1+i)^n}\,.} \tag{2}$$

Formula (2) shows that the principal required to yield an amount A in n periods at interest i is equal to $\frac{A}{(1+i)^n}$. In other words, if a sum P, equal to $\frac{A}{(1+i)^n}$, is invested *now* at interest i, it will produce an amount A after n periods. P is called the ***present value*** or ***present worth*** of A, due n periods hence. Values of the function $\frac{A}{(1+i)^n}$ for commonly used values of i and n are tabulated on pages 336-7.

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*An approximation for the length of time in which money doubles itself is given by the following rule: Divide 70 by the rate per cent per period to get the number of periods. Thus, in example 3, we have 70/2=35 periods or $8\frac{3}{4}$ years.

EXAMPLE 1. If a debt of \$400 falls due in 10 years, what is the present value of the debt if money is worth 5% per annum?

Since $A=\$400$, $n=10$, and $i=.05$, formula (2) gives $P=\frac{\$400}{1.05^{10}}=\$400(.613913)=\$245.57$, approximately. Thus an investment of \$245.57 at 5% now will produce \$400 in 10 years. Hence, the creditor should give a complete discharge for the debt for a cash payment of \$245.57.

The \$245.57 is called the present value of \$400, due in 10 years, with interest at 5% per annum. Thus, *the present value of a sum of money, due at a future date, is the principal which, if invested now at interest, would produce the given sum on the given date.*

EXAMPLE 2. Find the present value of a good promise to pay \$1000 in 6 years (*a*) if money is worth 4% compounded annually, (*b*) if money is worth 6% compounded semi-annually.

(*a*) From (2), we have

$$P=\frac{\$1000}{1.04^{6}}=\$1000\,(.790315)=\$790.32,$$

(*b*) From (2), we have

$$P=\frac{\$1000}{1.03^{12}}=\$1000\,(.701380)=\$701.38.$$

The difference, $A-P$, is sometimes referred to as the ***discount,****[*] or ***true discount.*** It is the amount that should be deducted from a future debt for a cash payment now. Thus in (*a*) the discount is \$1000—\$790.32=\$209.68, and in (*b*) the discount is \$1000—\$701.38=\$298.62.

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[*] This use of the word "discount" should not be confused with that employed in connection with the discounting of notes and bills of exchange. In this latter case, the discount is based on a *discount rate* which is quoted, whereas in our case, only an *interest* rate is involved. For example, if a bank quotes a discount rate of 6% per annum, the discount on a \$1000 note, due in 1 year, is \$60.

It is important to grasp clearly the distinction between amount and present value. The amount of \$100 in 5 years at 2% per annum is $\$100(1.02)^5$; this represents the *value at the end of 5 years* of \$100 invested now and allowed to accumulate with interest at 2% per annum. On the other hand, the present value of \$100, due in 5 years, when money is worth 2% per annum, is $\dfrac{\$100}{1.02^5}$; this is what a man should give *now* for a payment of \$100 which is to be made in 5 years, if he wishes to make 2% on his money.

EXERCISES

NOTE: Unless otherwise stated, the rate quoted is a rate per annum.

1. Write an expression for the amount of

(*a*) \$650 invested for 7 years at 3%, compounded annually;

(*b*) 2000 francs invested for 10 years at 4%, compounded semi-annually;

(*c*) £125 invested for 8 years at 6%, compounded quarterly.

2. Write an expression for the present value of

(*a*) 950 marks due in 5 years, if money is worth $3\frac{3}{4}$%, compounded annually;

(*b*) \$1000 due in 10 years, if money is worth 5%, compounded semi-annually;

(*c*) 500 pesos due in 6 years, if money is worth 2%, compounded quarterly.

3. Find the amount of \$5000 at the end of $6\frac{1}{2}$ years at 4% (*a*) compounded semi-annually, (*b*) compounded quarterly. In each case, state the interest received.

4. What rate, compounded annually, is equivalent to (*a*) 3% compounded semi-annually, (*b*) 4% compounded quarterly?

5. Find the present value of £2500 due at the end of 10 years, if money is worth 5% compounded (*a*) annually, (*b*) semi-annually, (*c*) quarterly.

6. In what time will money double itself at 6% compounded (*a*) annually, (*b*) semi-annually?

7. *A* holds *B*'s promise to pay him $7000 at the end of 7 years. If money is worth 5%, what discount should *A* allow *B* for a cash settlement?

8. What should a man pay for the right to collect $5000 at the end of 8 years if he wishes to make 4%, compounded semi-annually, on his investment?

9. Find the discount on $1200 due at the end of 6 years, if money is worth 3%.

10. In how many years will $100 amount to $141.60, if money is worth 2½%, compounded semi-annually?

11. In how many years will money triple itself if invested at 3½%, compounded annually?

12. A man buys a house by paying $2500 down, and agreeing to pay $1000 at the end of each year until 3 such payments have been made. If money is worth 4%, compounded semi-annually, find the equivalent cash price of the house.

13. Is 2½%, compounded semi-annually, a better investment rate than 2.6% compounded annually?

14. A man deposits $100 at the beginning of each year until 3 deposits have been made. If the bank in which the money is placed allows 2% interest, compounded semi-annually, what amount should be to his credit at the end of the third year?

15. A debt of £800 falls due at the end of 3 years, and another debt of £1000 falls due at the end of 10 years. If money is worth $3\frac{1}{2}\%$, compounded semi-annually, what single payment made at the end of 5 years would cancel both debts?

16. Show that the amount, A, of a sum, P, accumulated at *simple* interest at a fixed rate, i, is a linear function of the number of periods, n. Hence draw a graph showing the amount of 100 at 4% simple interest. Also, using formula (1) and plotting points corresponding to $n=5$, 10, 15, 20, and 25, draw a graph showing the amount of 100 at 4% compound interest.

78. Annuities. *Any sequence of payments made at periodic intervals is called an annuity.** The time which elapses between two consecutive payments is the ***period*** of the annuity. The word annuity connotes that the period is one year, and, while this is not always the case, it is common in practice; unless otherwise stated, it may be assumed that the period of an annuity is one year. For example, a bequest may provide a man with $100 at the end of each year for 10 years, the first payment to be made one year hence. This series of payments is described as an annuity of $100 which *begins now and runs for 10 years.* It is helpful to represent the payments on a diagram, thus:

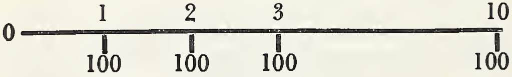

FIG. 27

The integer 3 above the line represents the *end* of the third year, and the number 100 below it represents the payment due at that time. Similarly, other payments may be represented so that the facts regarding the annuity are suggested by the figure.

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*An annuity paid for a definite number of years is sometimes called an *annuity certain,* in distinction from a *life annuity,* the payments of which are contingent upon the survival of an individual.

We shall confine our attention to annuities in which all payments are made on dates at which interest is compounded. For example, payments may be made on the first of July each year whereas interest is compounded on January 1st and July 1st. If an annuity begins *now*, the first payment comes one period hence, and the annuity runs from the beginning of the first period until the end of the last one. *If an annuity is **deferred** m periods, the first payment comes at the end of $m+1$ periods.* Thus, if an annuity is deferred 10 years, its first payment is due at the end of 11 years. An annuity whose payments continue without end is called a ***perpetuity.*** Perpetuities are often established for the payment of scholarships and bursaries.

Annuities give rise to problems of two types which are of basic importance in financial operations. We shall now discuss some examples of these problems.

78.1. Suppose that a man wishes to purchase the annuity represented in fig. 27. He goes to a financial house that can guarantee the payments, and learns that they will allow him 3% per annum on his money. What should he pay for these 10 annual payments of \$100, the first of which comes to him one year hence? Clearly, he should pay $\frac{\$100}{1.03}$ for the first payment, $\frac{\$100}{1.03^2}$ for the second, and so on. His total payment should be the sum of the present values of the individual payments of the annuity, or

$$\frac{\$100}{1.03}+\frac{\$100}{1.03^2}+\frac{\$100}{1.03^3}+\cdots+\frac{\$100}{1.03^{10}}.$$

This is a geometric series of 10 terms. Considering the terms as numbered from right to left and taking the sum, we get

$$\frac{\$100}{1.03^{10}}\left\{\frac{1.03^{10}-1}{1.03-1}\right\}=\frac{\$100}{.03}\left(1-\frac{1}{1.03^{10}}\right)=\$853.02.$$

This problem is that of *finding the present value of an annuity* which begins now and runs for 10 years, money being worth 3% per annum.

78.2. Consider now the problem which arises when a man who has been bequeathed the annuity represented in fig. 27, decides to bank the payments as they are made. If the bank pays 2% per annum on deposits, what will be the accumulated value of the savings just after the last payment is made? The first payment will be in the bank for 9 years, and will therefore amount to $\$100(1.02)^9$. Similarly, the second payment will amount to $\$100(1.02)^8$; the third will amount to $\$100(1.02)^7$; and so on. The last payment will draw no interest and so will add just \$100 to his savings. Hence, the required value of his savings is the sum of the amounts of the 10 payments, or

$$\$100\,(1.02)^9+\$100\,(1.02)^8+\cdots+\$100\,(1.02)+\$100.$$

Counting terms from right to left, we obtain for the sum of this geometric progression

$$\$100\left\{\frac{1.02^{10}-1}{1.02-1}\right\}=\frac{\$100}{.02}\,(.218994)=\$1095.0.$$

Here, we have found the *amount or the accumulated value of the 10-year annuity* immediately after the last payment is made, interest being at 2% per annum.

78.3. The two problems just discussed, namely, finding the present value and finding the amount of an annuity are of frequent occurrence. In each case, the summation of a geometric series is involved. The student may find it helpful in handling such problems to proceed as follows:

(*a*) represent the payments on a diagram as in fig. 27;

(*b*) decide whether the problem involves present value or amount;

(*c*) construct a geometric series, writing down the present value or amount (as the case may be) of the first two payments and the last payment;

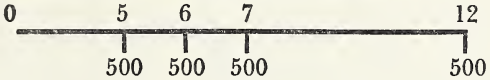
(*d*) find the sum of the resulting series by formula;

(*e*) use the tables to evaluate the formula.

79. Examples. The following examples illustrate further the use of the foregoing methods in common problems.

EXAMPLE 1. (*a*) If money is worth 4% per annum, find the present value of an 8-year annuity of \$500, deferred 4 years. (*b*) If each \$500 payment of an annuity similar to the one in (*a*) is immediately invested at $2\frac{1}{2}\%$ per annum, find the accumulated amount of the annuity just after the 8th payment is made.

(*a*) The annuity may be represented thus:



Therefore the present value, in dollars, is

$$P=\frac{500}{1.04^5}+\frac{500}{1.04^6}+\cdots+\frac{500}{1.04^{12}}$$

$$=\frac{500}{1.04^{12}}\left\{\frac{1.04^8-1}{1.04-1}\right\}=\frac{500}{.04}\left\{\frac{1}{1.04^4}-\frac{1}{1.04^{12}}\right\}$$

$$=12{,}500(.854804-.624597)=2877.6.$$

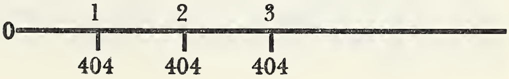
(*b*) Also, the amount, in dollars, is

$$A=500(1.025)^7+500(1.025)^6+\cdots+500(1.025)+500$$

$$=500\left\{\frac{1.025^8-1}{1.025-1}\right\}$$

$$=20{,}000(.218403)=4368.1.$$

EXAMPLE 2. Find the present value of a perpetuity of \$404 to begin now, money being worth 4% per annum, compounded semi-annually.



Since the present value of the perpetuity is the sum of the present values of the individual payments, we have

$$P = \frac{404}{1.02^2} + \frac{404}{1.02^4} + \frac{404}{1.02^6} + \cdots$$

$$= \frac{\frac{404}{1.02^2}}{1 - \frac{1}{1.02^2}} = \frac{404}{1.02^2 - 1} = 10{,}000.$$

Observe that the interest on \$10,000 for one year at 4%, compounded semi-annually, is \$404. Hence, the present value of the perpetuity is simply the principal, which, if invested at 4%, compounded semi-annually, would provide sufficient interest each year to meet the payment due.

EXERCISES

1. If money is worth $3\frac{1}{2}$% per annum, find the present value of (*a*) an annuity of \$1000 which begins now and runs for 10 years; (*b*) a 12-year annuity of 5000 francs, deferred 7 years.

2. What is the amount of an 8-year annuity of \$1200 just after the last payment is made, if money is worth $2\frac{1}{2}$%, compounded semi-annually?

3. What should be paid for a 15-year annuity of £500 by a man who wishes to make 4%, compounded semi-annually, on his money?

4. If money is worth 3%, what sum set aside now will provide for a \$300 annual scholarship, the payments of which are to begin in one year and to continue forever?

5. Payments of \$600 each are made at intervals of six months until 20 payments have been made. If each payment is immediately invested at 3%, compounded semi-annually, what is the amount of the annuity just after the 20th payment has been made?

6. A man purchases property by making a cash payment of \$5000 and agreeing to pay \$700 annually for the next 10 years. To what cash price is this equivalent if money is worth $4\frac{1}{2}\%$?

7. A father wishes to provide a sum of \$3000 to be available on his son's 18th birthday. He plans to do this by depositing a fixed sum on his son's first birthday and on each birthday thereafter, the last deposit being made on the 18th birthday. If the bank in which the money is deposited allows $2\frac{1}{2}\%$ interest, compounded semi-annually, what sum should the father deposit each time?

8. An investor has \$10,000 with which he wishes to purchase a 20-year annuity to begin now. If a financial house allows him $3\frac{1}{2}\%$ on his money, what should be the annual payment of the annuity?

9. At the present time, a man has \$9400. He invests this money at 4%, compounded semi-annually, and thereafter spends \$500 at the end of every six months. Show that his money will be used up by the end of the 12th year.

10. In settlement of a debt, a person makes two offers: (*a*) a cash payment of \$6000, and a payment of \$1800 at the end of 20 years; (*b*) an annual payment of \$500 at the end of each year for 20 years. If money is worth $4\frac{1}{2}\%$, which is the better offer for the creditor?

80. Bonds, debentures. Corporations and governments frequently borrow money through the issue of bonds or debentures. *A bond* is a promise, under seal, to pay a fixed sum, called the face value of the bond, at a definite future date, and, in addition, to pay interest at a specified rate on this sum.* The specified rate of interest is known as the ***bond rate,*** and the interest due is payable at regular periods. The dates of payment and the amounts payable are printed on a number of coupons which are attached to the bond. The bondholder clips these coupons and presents them for payment at the dates indicated.*

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*The discussion of this section applies to bonds which are redeemable at par. In some bonds, the sum payable at maturity, or the *redemption value,* is different from the face value, on which the interest is computed. If the redemption value exceeds the face value, the bond is said to be redeemable at a premium.

As an example, consider a \$1000, 4% bond, bearing half-yearly coupons and due exactly 20 years hence. This bond has a face value of \$1000, and this sum is payable at maturity. The bond rate is 4%, payable half-yearly. Hence, the bond will carry 40 coupons worth \$20 each, the first dated 6 months hence, and the last 20 years hence, at the due date, or date of maturity, of the bond. Therefore, the holder of the bond will receive

(*a*) an annuity of \$20 payable every six months, beginning now and running for 40 payments;

(*b*) \$1000 at the end of the 20 years.

The payments may be represented on a diagram, thus:

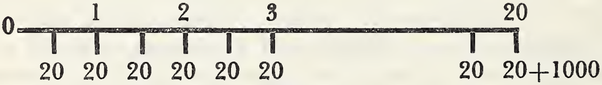

FIG. 28

It is to be noted that the interest payments are always calculated on the face value of the bond, the rate used being the bond rate. Such bonds as the ones we have discussed are called ***straight term*** or ***coupon*** bonds. We shall confine our attention to bonds of this type.

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*Some bonds are registered as to principal and interest in the name of the owner. This is a protection against theft or loss since such bonds are of value only to the registered holder. The interest is mailed by cheque direct to the owner. The advantage of safety in the ownership of a registered bond is partially offset by the inconvenience of transferring ownership. Most bonds are coupon bonds.

80.1. In practice, the words "bond" and "debenture" are somewhat loosely interchanged. There is this difference, however, that corporation bonds are usually secured by a mortgage on the physical plant, etc., while debentures are generally unsecured, except by the integrity of the corporation. In other words, a debenture holder's position is often that of a creditor whose claims take precedence over those of preferred and common stock holders. Government and municipal bonds are, as a rule, simply promises to pay under seal, and no mortgage is necessary to secure them. From the point of view of the problems to which they give rise, there is no essential difference between bonds and debentures.

81. Problems in the buying and selling of bonds. When an issue of bonds is placed on the market by a corporation, municipality, or government, it is generally bought by one or more bond houses. The bond house decides upon a "bid" for the bonds by first estimating the rate of interest which the bond should yield to an investor. This so-called ***yield rate*** depends upon the security behind the bond, the ability of the issuer to meet coupon payments, the ease with which the bond can be converted into cash, etc. It is important to note that, in this decision as to the yield rate, the bond rate or value of the coupon plays no part. For example, a financial house, after proper consideration, decides what rate of interest an investment should yield; on the basis of this yield, they calculate their "bid" for the bond, that is, the price which should be paid for the bond in order to realize the desired return. The determination of this price is based on the yield rate, the bond rate, and the term or number of years to maturity. In other words, the price is a function of the yield rate, the bond rate, and the term, which are independent variables.

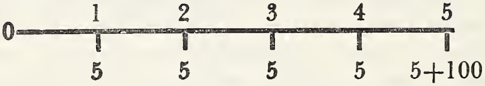
Thus, when the bonds are sold by the bond house to the public, they are offered to the individual investor at a price which is already fixed (it may fluctuate slightly, with business conditions, from day to day). The problem which is of interest to the investor is that of determining the yield rate which now depends upon the bond rate, the market price, and the term of the bond.

When the market price of a bond is below the face value, the bond is said to be quoted at a ***discount;*** the yield rate is then higher than the bond rate. Similarly, when the market price is above face value, the bond is quoted at a ***premium,*** and the yield rate is lower than the bond rate.

In the following examples, the two problems mentioned above are considered: first, that of the bond house in determining the price of the bond when the yield rate, the bond rate, and the term are given; second, that of the investor in determining the yield rate when the bond rate, the price, and the term are given.

EXAMPLE 1. Determine the price which should be paid for a \$100, 5% bond, bearing yearly coupons and having 5 years to run, (*a*) if the investment is to yield 4%, (*b*) if it is to yield 6%.

(*a*) The purchaser will receive an annuity of \$5 for 5 years, and \$100 redemption value at the end of the 5th year, as represented



The present value of these payments to a purchaser who wants the investment to yield 4% is given by

$$\begin{aligned}
P &= \frac{5}{1.04} + \frac{5}{1.04^2} + \cdots + \frac{5}{1.04^5} + \frac{100}{1.04^5} \\
&= \frac{5}{1.04^5}\left\{\frac{1.04^5-1}{.04}\right\} + \frac{100}{1.04^5} \\
&= \frac{5}{.04}\left\{1-\frac{1}{1.04^5}\right\} + 100\left\{\frac{1}{1.04^5}\right\} \\
&= 125\,(.178073) + 100\,(.821927) \\
&= 22.259 + 82.193 = 104.452.
\end{aligned}$$

Therefore, the price is \$104.45 and the bond is bought at a premium of \$4.45. Since the value of the bond at maturity is exactly \$100, this premium must be gradually extinguished, or *amortized,** during the 5 years so that the value of the investment shrinks to \$100 at the redemption date. The value of the bond on the purchaser's books is therefore decreased each year in the manner illustrated in the following table.

SCHEDULE OF BOOK VALUES

(Amortization Schedule)

Year	Book value at first of year	4% yield on book value	Coupon: 5% on face value	Decrease in book value	Book value at end of year
1	104.452	4.178	5.00	.822	103.630
2	103.630	4.145	5.00	.855	102.775
3	102.775	4.111	5.00	.889	101.886
4	101.886	4.075	5.00	.925	100.961
5	100.961	4.038	5.00	.962	99.999

(*b*) In a similar manner, for a 6% yield, we have

$$P=\frac{5}{1.06}+\frac{5}{1.06^2}+\cdots+\frac{5}{1.06^5}+\frac{100}{1.06^5}$$

$$=\frac{5}{1.06^5}\left\{\frac{1.06^5-1}{.06}\right\}+\frac{100}{1.06^5}$$

$$=\frac{5}{.06}\left\{1-\frac{1}{1.06^5}\right\}+100(.747258)$$

$$=21.062+74.726=95.788.$$

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*Literally "killed off".

Here, the bond is bought at a discount of \$4.212. This discount is *accumulated* over the 5-year period, so that the value of the investment on the purchaser's books grows to \$100 at maturity. This accumulation is brought about as shown in the following table.

SCHEDULE OF BOOK VALUES

(*Accumulation Schedule*)

Year	Book value at first of year	6% yield on book value	Coupon: 5% on face value	Increase in book value	Book value at end of year
1	95.788	5.747	5.00	.747	96.535
2	96.535	5.792	5.00	.792	97.327
3	97.327	5.840	5.00	.840	98.167
4	98.167	5.890	5.00	.890	99.057
5	99.057	5.943	5.00	.943	100.000

81.1. The example just considered is typical of the problems which confront a bond house; the price of a bond was determined when the yield rate, the bond rate, and the term were given. We consider next the related problem which is of interest to investors—that of determining the yield rate from the bond rate, the term, and the quoted market price.

One method of handling this problem is to estimate the yield rate, and then find the corresponding market price. If this price is compared with the actual market price, the estimate can be revised and the process repeated. When two market prices found from estimated yields are such that the actual market price lies between them in magnitude, interpolation may be used.

A formula which gives a reasonably good approximation for the yield rate can be obtained as follows. Suppose a bond, with n periods to run, having a face value F, and bearing coupons worth c each, sells at a premium of p above its face value. Since the original investment, $F+p$, is reduced to F at maturity, we may consider the average investment to be $\frac{2F+p}{2}$, or $F+\frac{p}{2}$. Also, each coupon payment c provides partial amortization of p, as well as income for one period. Since there are n periods, the average amortization per period is $\frac{p}{n}$, and the average income, $c-\frac{p}{n}$. If we consider the approximate yield rate, y, to be the average income divided by the average investment, we have

$$y=\frac{c-\frac{p}{n}}{F+\frac{p}{2}}.$$

EXAMPLE 2. A $100, 5% bond, bearing yearly coupons and having 5 years to run, is bought at $104.45. What is the yield rate?

Here, $F=100$, $c=5$, $n=5$, and $p=4.45$. Therefore, we have

$$y=\frac{5-.89}{100+2.23}=.040.$$

This gives an estimate of 4% for the yield rate. This rate may now be checked by finding the price at which a $100, 5-year, 5% bond should be bought in order to yield 4% on the investment. This has already been done in example 1 and the price found checks with that given here.

Similarly, if a bond, having a face value F and n periods to run, and bearing coupons worth *c* each, sells at a discount *d*, we obtain for the approximate yield rate

$$y = \frac{c + \frac{d}{n}}{F - \frac{d}{2}}.$$

82. Mortgages. Private individuals and organizations often borrow money by means of *mortgages* on their possessions. A mortgage is a signed promise to pay a sum of money at a future date and interest thereon at a specified rate in the meantime. This promise is given to the lender and is secured by property, the title to which may go to the lender, if the borrower fails to meet his obligations. The payments on a mortgage are similar to those required for a bond. In fact, fig. 28 may equally well be used to represent a $1000 mortgage, bearing interest at 4% payable half-yearly, and having 20 years to run. In other words, the payments consist of an annuity plus the return of the loan at the end of the term. Problems which arise with mortgages are similar in nature and treatment to those with bonds.

EXERCISES

NOTE: Unless otherwise stated, the rate quoted is a rate per annum.

1. What should an investor pay for a $1000, 3% bond, bearing yearly coupons and due in exactly 6 years, if he wishes a yield of 4% on the investment?

2. For the bond of the preceding problem, make a table similar to that of section 81, showing the schedule of book values.

3. An issue of 5-year, 4% bonds, bearing yearly coupons, is sold to give a yield rate of 3%. What is the price of a $500 bond?

4. For the bond of the preceding problem, make a table showing the schedule of book values.

5. A $7000 mortgage, bearing interest at 6%, has exactly 10 years to run. What should an investor pay for this mortgage if he wishes to make 4%, compounded semi-annually?

6. A municipality desires to raise $100,000 by issuing a series of 20-year, 4% debentures, bearing half-yearly coupons. At what price should these debentures be quoted to yield an investor $3\frac{1}{2}$%, compounded semi-annually?

7. A 20-year, 5% municipal bond with half-yearly coupons is offered at 113. Is this a good investment for a man who is satisfied to make 4%, compounded semi-annually, on his money?

8. A $1000, 15-year, 3% bond bears half-yearly coupons and is redeemable at a premium of 10%. What should be the price of this bond in order to yield the investor $2\frac{1}{2}$%, compounded semi-annually?

9. If money is worth 3%, compounded semi-annually, find the difference between the present value of a $1000, 10-year, 4% bond, bearing half-yearly coupons, and that of a $1200, 15-year, $3\frac{1}{2}$% bond, bearing half-yearly coupons.

10. A mortgage for $4000 is payable in 8 years and bears interest at 6%, payable quarterly. What is the present value of this mortgage to an investor who wishes to make 4%, compounded quarterly, on his money?

11. On January 1, 1938, an investor purchases at 98 a $100, $3\frac{1}{2}$% bond, bearing yearly coupons. At what price should he sell the bond on January 1, 1950, in order to realize 4% per annum on his investment?

12. A \$100, 5% bond, bearing yearly coupons and redeemable at par in exactly 15 years, is sold for \$104.

(*a*) Estimate the yield rate by computing the price of this bond for a yield of $4\frac{1}{2}\%$, and then interpolating.

(*b*) Check the estimate obtained in part (*a*) by using the approximate formula of sec. 81.1.

83. Payment of debts. Most of the borrowing by corporations and governments is done through the issue of bonds or debentures. How are these debts retired? In common practice, there are two plans for extinguishing debts: (*a*) ***the sinking fund plan,*** (*b*) ***the amortization or instalment plan.***

For business reasons into which we shall not enter, one of these plans is usually preferable to the other in a given situation. The difference between the two lies in the manner in which the principal of the debt is paid. We shall discuss the two plans separately with the aid of examples.

83.1. The sinking fund plan. A sinking fund is an annuity set aside to accumulate for the purpose of retiring a debt due in the future. To retire bonded, or other, indebtedness by the sinking fund plan, it is necessary to provide (i) sufficient money to pay the interest or coupon payments when due, and (ii) periodic sinking fund payments large enough, when accumulated, to retire the debt at maturity. We shall assume that the payments which constitute the sinking fund are equal, and that a payment is placed in the fund at the end of each interest period.

EXAMPLE 1. A municipality has just sold a \$100,000 issue of 20-year, 5% sinking fund debentures, bearing yearly coupons. If the sinking fund can be invested at 4%, what amount of money must be raised each year to retire the issue at maturity?

The coupon payments each year are 5% of \$100,000 or \$5000. In addition to this, let $\$x$ be set aside at the end of each year for 20 years to form a sinking fund. This sinking fund can be accumulated at 4%, and the payments of $x must be such that the amount of the annuity, just after the last payment is made, is $100,000. Therefore, we have

$$x(1.04)^{19}+x(1.04)^{18}+\cdots+x(1.04)+x=100{,}000.$$

or

$$\frac{x\,(1.04^{20}-1)}{.04}=100{,}000.$$

Hence

$$x=\frac{4000}{1.04^{20}-1}=3358.2.$$

Thus, the total amount required each year is

$$\$5000+\$3358.2=\$8358.2.$$

83.2. The amortization or instalment plan. Under the sinking fund plan, the interest on the whole debt is paid yearly; the principal of the debt remains unchanged until the end of the term, while assets are accumulated in a sinking fund sufficient to *retire the principal in a single payment* at that time. On the other hand, under the instalment plan, the entire debt, both principal and interest, is retired by a number of equal, annual instalments. In this case, interest each year is paid only on the outstanding principal, and *the principal is reduced each year* until by the end of the term it is wiped out. (See table, p. 303.)

EXAMPLE 2. A city issues 4% instalment debentures to the value of $10,000. If these are to be payable, principal and interest, in 10 equal, annual instalments, what amount must be included in the city's budget each year to handle the issue?

Let the yearly amount necessary be $x. Then the present value of these annual instalments must equal the present value of the debt. Hence, the 10 annual payments of $x, beginning at the end of the first year, have a present value of \$10,000. Since the debenture rate is 4%, we have

$$10{,}000 = \frac{x}{1.04} + \frac{x}{1.04^2} + \cdots + \frac{x}{1.04^{10}}$$

$$= \frac{x}{1.04^{10}}\left\{\frac{1.04^{10}-1}{.04}\right\}$$

$$= \frac{x}{.04}\left\{1 - \frac{1}{1.04^{10}}\right\}$$

Therefore

$$x = \frac{400}{.3244358} = 1232.91.$$

Hence, by budgeting for \$1232.91 each year, the city could retire the issue. According to this plan, a certain amount of the bonds would be retired, or amortized, each year. The table (p. 303) shows the precise nature of this retirement.

It is seen that some of these bonds will have irregular face values. For example, bonds with total face value \$832.91 are retired at the end of the first year. This is usually handled by issuing two 1-year bonds, one for \$500 and the other for \$332.91. These bonds each carry one coupon which is for \$20 in the first case, and for \$13.32 in the second. Similarly, bonds with face value \$866.23 are retired at the end of the second year, and so on. Such bonds, issued by municipalities with good credit, are more readily purchased by insurance companies, banks, and trust companies than by the general public. Hence, bonds issued under the sinking fund plan with face values of \$100, \$500, \$1000, etc., usually command a somewhat wider market.

The essential difference between the two plans, therefore, is in the manner in which the principal is extinguished. In the sinking fund plan, the principal is paid at maturity by a single payment and interest is paid throughout on the entire principal; in the instalment plan, the principal is paid in instalments, and interest is paid only on the principal outstanding. It will be

Retirement Schedule

Year	Debt at first of year	Interest due at 4%	Total annual payment	Amount of issue retired	Debt at end of year
1	10,000.00	400.00	1232.91	832.91	9167.09
2	9167.09	366.68	1232.91	866.23	8300.86
3	8300.86	332.03	1232.91	900.88	7399.98
4	7399.98	296.00	1232.91	936.91	6463.07
5	6463.07	258.52	1232.91	974.39	5488.68
6	5488.68	219.55	1232.91	1013.36	4475.32
7	4475.32	179.01	1232.91	1053.90	3421.42
8	3421.42	136.86	1232.91	1096.05	2325.37
9	2325.37	93.02	1232.91	1139.89	1185.48
10	1185.48	47.42	1232.91	1185.49	0000.00

noticed that, if the sinking fund earns the same rate of interest as is paid on the bonds, then there is no difference, under the two plans, in the amounts of money that must be raised each year to retire a debt.

83.3. The two plans for extinguishing debts have, in the foregoing examples, been applied to the retiring of bond issues. However, either of these plans is applicable in the same manner to the retirement of debts arising in some other way. The sinking fund method is also used to provide money for future replacements of equipment in factories, railways, etc. If the life of a piece of equipment and the future cost of replacing it are known, the annual payment required for the sinking fund (or ***depreciation fund***, as it is often called) is calculated as in sec. 83.1.

EXERCISES

1. A corporation sells a $70,000 issue of 30-year, 4% bonds, bearing yearly coupons. These bonds are to be retired under the sinking fund plan. If the sinking fund can be invested at 3% per annum, what amount of money must be raised each year to retire the issue at maturity?

2. A city borrows $50,000 by an issue of 3% instalment debentures. The entire issue, both principal and interest, is to be retired in 10 equal, annual instalments. What amount of taxes must the city raise each year to extinguish this debt?

3. For the preceding problem, make a table as in sec. 83.2 to show the precise nature of the retirement schedule.

4. A street car costs $60,000 and has an estimated life of 20 years. What amount should be deposited in a depreciation fund at the end of each six months in order to replace the car when worn out, if the car has a final scrap value of $500 and the fund can be invested at 3½% per annum, compounded half-yearly?

5. A city issues $10,000 worth of 15-year, 3½% bonds, bearing half-yearly coupons. The issue is to be retired under the sinking fund plan, and equal payments are set aside for the fund at the end of each six months. If the sinking fund accumulates at 3%, compounded semi-annually, what sum must be raised every six months?

6. A debt of $6000, bearing interest at 5%, compounded semi-annually, is to be retired, both principal and interest, in 10 equal, semi-annual payments made on the dates at which the interest is compounded. Find the amount of money required

for each half-yearly payment. How much of the principal will have been retired at the end of one year?

7. A municipality needs to borrow $200,000 for improvements. This amount is to be raised through an issue of 25-year, $4\frac{1}{2}\%$ bonds, bearing yearly coupons. Two plans for retiring the issue are suggested as follows:

(*a*) the sinking fund plan, under which the sinking fund would be invested at 4% per annum;

(*b*) the instalment plan, under which the debt, both principal and interest, would be paid in 25 equal, annual instalments.

What sum would the municipality have to raise yearly under each plan?

EXERCISES ON CHAPTER X

A

1. Find the amount of 25,000 francs at the end of 10 years if the money is accumulated at 4% per annum (*a*) compounded annually, (*b*) compounded semi-annually, (*c*) compounded quarterly.

2. What is the present value of $10,000 due at the end of 8 years if money is worth 3% per annum (*a*) compounded annually, (*b*) compounded semi-annually, (*c*) compounded quarterly?

3. What rate of interest compounded annually is equivalent to (*a*) 4% compounded semi-annually, (*b*) 5% compounded quarterly?

4. If money is worth 5%, find the present value of an annuity of £400, deferred for 4 years and having 10 years to run.

5. If the payments of the annuity of exercise 4 are invested upon receipt at 3%, compounded semi-annually, find the amount of the annuity immediately after the last payment is made.

6. A \$1000, $3\frac{1}{2}$% bond, bearing half-yearly coupons and having 12 years to run, was purchased by an investor to yield 4% per annum, compounded semi-annually. What was the price paid?

7. A man wishes to redeem a mortgage of \$8000 bearing interest at 6% per annum, payable half-yearly, and having 6 years to run. What is the cash value of this mortgage if money is worth 4% per annum, compounded semi-annually?

8. A \$100, $3\frac{1}{2}$% bond, bearing yearly coupons and having 4 years to run, is purchased to yield 3% per annum. Find the price paid and make a table showing the yearly decrease in the book value.

9. A debt of \$1000, bearing interest at 5% per annum, falls due in 8 years. The debtor wishes to provide for the principal when due by creating a sinking fund which accumulates at $3\frac{1}{2}$% per annum. Find (*a*) the annual sinking fund payment, (*b*) the total amount needed each year to retire the debt under this plan.

10. If the debt of exercise 9 is retired, both principal and interest, in 8 equal, annual instalments payable when interest payments are due, find the amount of each instalment.

11. For the preceding problem, construct a table showing the retirement schedule. How much of the principal remains unpaid after the fifth instalment has been paid?

B

12. A man pays \$1100 for an automobile which he plans to use for 4 years and then turn in on a similar model. To allow for depreciation, the owner will deposit a fixed amount in a

bank at the end of every six months. If the bank allows interest at 2%, payable half-yearly, on the dates at which the deposits are made, and if the turn-in value of the car at the end of 4 years is $500, how large should these semi-annual deposits be in order to provide for the purchase of an $1100 automobile at the end of 4 years?

13. An investor buys a 15-year annuity, deferred 5 years, for $25,000. If money is worth $3\frac{1}{2}$% per annum, what is the annual payment of the annuity?

14. What is the present value at 5% per annum of a payment of $50,000 every 25 years forever, the first payment to be made one year hence?

15. A city has just sold a $250,000 issue of 20-year, 4% sinking fund bonds, bearing half-yearly coupons. If the sinking fund is accumulated at 3%, compounded semi-annually, and if payments to the fund are made at the end of each six month interest period, what amount of money must be available every half-year in order to extinguish the debt under this plan?

16. A corporation issues 5% instalment debentures to the value of $100,000. These debentures bear half-yearly coupons and are to be retired, both principal and interest, by 40 equal, semi-annual instalments. Find the amount of money needed every six months to retire the issue under this plan. What amount of the bonds will be retired by the payment of the first two instalments?

17. An investor deposits $1000 at the end of each year for 20 years with a view to purchasing a 10-year annuity with his accumulated savings. The first payment of the annuity is to come one year after he makes his last $1000 deposit. If money is worth 4% per annum throughout, find the annual payment of the annuity.

18. A mine yields a net annual income of \$10,000. If the resources of the mine will be exhausted at the end of 25 years, find, correct to 4 figures, the price which should be paid for the mine in order to make 7% per annum on the investment.

19. A man pays \$41.60 to an insurance company at the beginning of each year for a 20-year endowment policy of \$1000. If the man survives, he will receive \$1000 at the end of 20 years. What sum, invested at the beginning of each year for 20 years, would amount to \$1000 at the end of 20 years if money is worth 4% per annum? If the provision for the payment of \$1000 at the end of 20 years were deleted from the policy, what would be the annual premium?

20. Under a 20-payment life policy, a man paid a premium of \$25.40 at the beginning of each year for 20 years and his beneficiary received \$1000 on his death which occurred 25 years after he took out the policy. If the premiums had been deposited in a bank to accumulate at $2\frac{1}{2}$% per annum, how would the value of the man's estate have been altered?

21. The annual fees for membership in a club are \$100 payable at the beginning of the year, and a life membership costs \$2100. If money is worth 3%, how long must a member live to make a life membership worth while?

22. A man receives a bequest of \$5000 which he deposits in a bank paying 2% per annum, compounded semi-annually. If he withdraws \$250 at the end of every six months thereafter, how much money remains to his credit just after the 20th withdrawal is made?

C

23. Dominion of Canada bonds, payable in 12 years and bearing 4% half-yearly coupons, are offered for sale at 108. Estimate, to a tenth of one per cent, the yield rate to which this price corresponds.

24. A taxicab company finds that its cars earn on the average \$800 a year and that the cars must be replaced at the end of every 3 years. If the cars have a turn-in value of \$200 when replaced, what should the company pay for its new cars in order to make 10%, compounded semi-annually, on its investment?

25. A \$1000, 4% debenture, bearing half-yearly coupons, matures in 6 years. A purchaser plans to put his semi-annual interest receipts in a bank which allows 2% per annum, compounded semi-annually. What price should he pay for the bond if he wishes his entire investment to yield 3% per annum, compounded semi-annually?

26. A \$100, 10-year, 4% bond, bearing yearly coupons, sells for \$97.50. Estimate the yield rate to a tenth of one per cent.

27. An investor pays \$1000 for a 10-year annuity of \$133.29, the first payment to be made one year hence. Of each payment of \$133.29, he retains \$1000i as interest, and sets aside the balance, \$133.29–\$1000i, to accumulate at 4% per annum. If these sums which are set aside amount to \$1000 at the end of 10 years, find the rate i.

28. A loan of \$6000, bearing interest at 5½% per annum, is to be repaid, both principal and interest, in 12 equal, annual instalments. An investor purchases the note just after the sixth instalment has been paid. If he is making 4% on his investment, what is the purchase price?

29. Two men find that they have accumulated the same amount, the first by saving \$500 a year for 20 years, the other by saving \$1140 a year for 10 years. If both receive the same rate of interest and the day of comparison is that on which each comes in with his last deposit, find the rate of interest.

30. A \$500, 5% bond, bearing yearly coupons, matures in 10 years. What price should an investor pay for this bond (*a*) to realize 4% per annum on his investment, (*b*) to realize 3½% per annum. If the bond is purchased for \$550, find the yield rate.

31. A \$1000, 4% bond bears half-yearly coupons and is redeemable at 10% premium in 9 years. If the bond is quoted at 115, show that this is a good price for an investor who wishes to make 3% per annum, compounded semi-annually.

32. A \$1000, 5% bond, bearing yearly coupons and having 3 years to run, sells at 106. If money can be invested at 4% per annum, should the holder of such a bond keep it until maturity or sell it and re-invest? If the holder keeps the bond until maturity, what yield rate does he receive on his investment during the last 3 years?

33. In connection with a loan, a man gave his note for \$60 to be paid in six monthly instalments of \$10, the first instalment to be paid one month hence. He received \$60 less \$5 for interest and service charges. Assuming that interest is simple, estimate the rate of interest per cent per month to which this \$5 charge is equivalent. To what rate per cent per annum is this charge equivalent?

APPENDIX I

SOME PROPERTIES OF REAL NUMBERS

A complete discussion of real numbers is out of place in this book. We shall take for granted the existence of real numbers, positive and negative, rational and irrational. The basic operations in which numbers are used are addition, subtraction, multiplication, and division, which are called the four rational operations. What are the rules which govern these operations? It is well known that, in geometry, the demonstrations are based on certain *assumptions* or *axioms*. It is not quite so obvious that algebra too is based on assumptions. The performing of the fundamental operations becomes so much a matter of habit that it requires a mental effort for us to realize that the laws governing them are logically only assumptions. Any one of these could be replaced by a contrary assumption. The choice of assumptions depends upon the fact that the mathematics which results from them has useful applications in science and practical affairs. For this reason, the assumptions appear to us as reasonable while any other assumptions do not seem to fit in with our experience. The object of this section is to state the assumptions which form the basis of the four rational operations.

It is assumed that addition of two or more real numbers is always possible and obeys the following two laws:

I. *The commutative law.* This states that

$$a+b=b+a.$$

In other words, the order of the terms in a sum is immaterial.

II. *The associative law.* This states that

$$a+(b+c)=(a+b)+c.$$

When three numbers are added, it makes no difference which two are added first.

The operation inverse to addition is subtraction. The assumption here is that, if a and b are any two numbers, there exists a number c such that $a+c=b$. The number c is denoted by $b-a$ and the operation by which it is obtained is called subtraction. Thus subtraction of two numbers is always possible and leads to only one result.

It is assumed that multiplication of two or more numbers is always possible and obeys the following three laws:

I. *The commutative law.* This states that

$$ab=ba.$$

The order of the factors in a product is immaterial; in other words, a multiplied by b is the same as b multiplied by a.

II. *The associative law.* This states that

$$a(bc)=(ab)c.$$

In taking the product of three numbers, it makes no difference which two are multiplied first.

III. *The distributive law.* This states that

$$a(b+c)=ab+ac.$$

In taking the product of one number by the sum of two others, we may perform an addition followed by a multiplication, as indicated in the left member, or two multiplications followed by an addition, as indicated in the right member.

The operation inverse to multiplication is division. The assumption here is that, if a and b are two numbers of which $b\neq 0$, there exists a number c such as $a=bc$. The number c is denoted by $a\div b$ or $\frac{a}{b}$, and the operation by which it is obtained is called division, the number b being the divisor. Thus division is always possible except when the divisor is zero.

Another assumption which is much used in elementary algebra relates to the vanishing of a product. If

$$ab=0,$$

then either $a=0$ or $b=0$. A product is zero if, and only if, one of its factors is zero.

APPENDIX II

COMPLEX NUMBERS

In reading the following discussion of complex numbers, the student should put aside any preconceptions he may have regarding the meaning of these numbers. Assuming that real numbers and their rules of operation are known, we proceed to define a complex number. The complex number shall consist of a pair of real numbers, a and b, which shall be used according to certain rules to be laid down. We make no presumption, in advance of these rules, regarding the manner in which the two real numbers make up the complex number; hence we shall, at first, employ the unexpressive symbol (a, b). The definition will now be made by setting up rules of operation which give definite meaning to the "complex number" (a, b). These rules of operation are defined as follows:

1. **Equality.** Two complex numbers (a, b) and (c, d) are said to be equal if, and only if, $a=c$ and $b=d$. That is

$$(a, b)=(c, d) \text{ if, and only if, } a=c \text{ and } b=d.$$

The complex number $(0, 0)$ is called zero and is denoted, for short, by 0.

2. **Addition.** From two complex numbers (a, b) and (c, d), we can form a third number $(a+c, b+d)$. The last number is called the *sum* of the other two, and the operation by which it is obtained is called addition. In symbols,

$$(a, b)+(c, d)=(a+c, b+d).$$

From this definition, it follows that the commutative and associative laws of addition hold for complex numbers. We give the proof for the commutative law.

By definition, $(a, b)+(c, d)=(a+c, b+d)$,

and $(c, d)+(a, b)=(c+a, d+b)$.

Since a, b, c, and d are real numbers,

$$a+c=c+a, \text{ and } b+d=d+b.$$

Hence $(a+c, b+d)=(c+a, d+b)$,

and therefore $(a, b)+(c, d)=(c, d)+(a, b)$.

2.1. Subtraction. The following result is a consequence of the definitions of secs. 1 and 2. If (a, b) and (c, d) are given complex numbers, then a third number (x, y) exists such that

$$(a, b)+(x, y)=(c, d).$$

For, in order that this be true, it is necessary and sufficient that

$$(a+x, b+y)=(c, d).$$

Hence, from sec. 1, $a+x=c$

and $b+y=d$,

whence $x=c-a$ and $y=d-b$.

The complex number $(c-a, d-b)$ is said to be obtained from (a, b) and (c, d) by *subtraction*. In symbols,

$$(a, b)-(c, d)=(a-c, b-d).$$

Thus, subtraction of two complex numbers is always possible and leads to only one result.

3. Multiplication. From two complex numbers (a, b) and (c, d), we now form the number $(ac-bd, ad+bc)$. This last number is called the product of the first two and the operation of obtaining it is called multiplication. In symbols,

$$(a, b)\times(c, d)=(ac-bd, ad+bc).$$

From this definition, it can be proved that the commutative, associative, and distributive laws of multiplication hold for complex numbers. Take, for example, the associative law. We wish to prove that

$$(a, b)\times\{(c, d)\times(e, f)\}=\{(a, b)\times(c, d)\}\times(e, f).$$

We have

$$\begin{aligned}(a, b)\times\{(c, d)\times(e, f)\} &= (a, b)\times(ce-df,\ cf+de)\\ &= (ace-adf-bcf-bde,\ acf+ade+bce-bdf).\end{aligned}$$

Also

$$\begin{aligned}\{(a, b)\times(c, d)\}\times(e, f) &= (ac-bd,\ ad+bc)\times(e, f)\\ &= (ace-bde-adf-bcf,\ acf-bdf+ade+bce).\end{aligned}$$

Since the two results are equal, the theorem is proved. Similarly, it can be shown that the other two laws hold.

3.1. Division. The following result is a consequence of the definitions of secs. 1 and 3. If (a, b) and (c, d) are given complex numbers, of which (c, d) is not the number 0, then a third number (x, y) exists such that

$$(a, b)=(c, d)\times(x, y).$$

For, in order that this be true, it is necessary and sufficient that

$$(a, b)=(cx-dy,\ dx+cy).$$

From sec. 1, this is true if, and only if,

$$cx-dy=a,$$
$$dx+cy=b.$$

It follows that $$(c^2+d^2)x=ac+bd.$$

Since $(c, d)\neq(0, 0)$, therefore c and d are not both 0 and $c^2+d^2\neq0$. Hence

$$x=\frac{ac+bd}{c^2+d^2}.$$

Similarly,

$$y=\frac{bc-ad}{c^2+d^2}.$$

The complex number $\left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$ is said to be obtained by *division* of (a, b) by (c, d); (c, d) is called the *divisor*. In symbols,

$$\frac{(a, b)}{(c, d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right).$$

Thus, division of two complex numbers is always possible except when the divisor is zero. Also the operation gives a unique result whenever it is possible. Division by zero is not defined.

4. Relation of complex numbers to real numbers. We now investigate more definitely the relation of complex numbers to real numbers. Any complex number can be expressed as the sum of two others, thus:

$$(a, b) = (a, 0) + (0, b).$$

It is therefore sufficient to investigate numbers of these two types.

4.1. Complex numbers of the type $(a, 0)$. Numbers of this type form a class of numbers for which the rules of operation reduce to the following:

$$(a, 0) + (a', 0) = (a+a', 0),$$
$$(a, 0) - (a', 0) = (a-a', 0),$$
$$(a, 0) \times (a', 0) = (aa', 0),$$
$$(a, 0) \div (a', 0) = \left(\frac{a}{a'}, 0\right), \text{ if } a' \neq 0.$$

In these operations, the second number in each bracket, 0, plays no part. The operations are all performed by operating, with the rules for real numbers, on the numbers a and a'. Since any calculation involving a number $(a, 0)$ can be carried out by replacing this number by the real number a, we shall agree to identify the number $(a, 0)$ with the real number a, so that

$$(a, 0)=a.$$

With this agreement, the class of real numbers becomes a subclass of all complex numbers, namely those in which the second number in each pair is zero.

4.2. Complex numbers of the type $(0, b)$. It is easily verified that

$$(0, b)=(b, 0)\times(0, 1)=b\times(0, 1).$$

There remains to be considered the single number $(0, 1)$. Its relation to real numbers can be seen as follows. By the rule for multiplication,

$$(0, 1)\times(0, 1)=(-1, 0)=-1.$$

Hence $(0, 1)$ is a complex number with the property that its square is -1. This property is possessed by no real number. Let us denote the number $(0, 1)$ by the letter i. Then

$$(0, b)=b\times i=bi.$$

Finally

$$(a, b)=(a, 0)+(0, b)=a+bi.$$

The foregoing definitions have been so framed that complex numbers can be employed in the same way as real numbers, with the added rule that $i^2=-1$. Thus the rule for multiplication (sec. 3), which was written

$$(a, b)\times(c, d)=(ac-bd, ad+bc),$$

may now be written

$$(a+bi)(c+di)=ac-bd+(ad+bc)i,$$

a result which can be obtained by the ordinary process of multiplication. Similarly, the rule for division can now be written

$$\frac{a+bi}{c+di}=\frac{ac+bd+(bc-ad)i}{c^2+d^2}.$$

This result is most conveniently obtained, in practice, by multiplying the numerator and the denominator of the given fraction by $c-di$.

ADDITIONAL EXERCISES

The following problems are designed to give further practice to students interested in algebra. They are arranged in 10 sets, each of which contains 10 problems diversified in character.

I

1. If r and s are the roots of $x^2-pqx+p+q=0$, show that p and q are the roots of $x^2-rsx+r+s=0$.

2. Given $v=\dfrac{12t}{2+s}$, what is the effect on v of (a) increasing t while s is constant, (b) increasing s while t is constant? In (a), if t increases uniformly, does v change at a constant rate? In (b), if s increases uniformly, does v change at a constant rate? If not, show when v changes rapidly and when slowly.

3. From the inequality $(x-y)^2\geqq 0$, which holds for real numbers, what conclusion can be drawn regarding the fraction

$$\frac{x^2+y^2}{xy}?$$

4. (a) For what values of x is $(x-1)(2x-3)(2x+3)$ greater than zero?

(b) For what values of x is $(x-3)(2x+1)(3x-1)$ less than zero?

5. By inspection of the equation, find the two roots of

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}=x^2.$$

6. A number of spheres are piled in a pyramidal pile of which the bottom layer has the form of an equilateral triangle with n spheres to a side. If the pile is complete, so that the top layer consists of one sphere, find the total number of spheres in the pile.

7. Prove that, if a is a positive number,

$$\sqrt{a\sqrt{a\sqrt{a\sqrt{a\cdots}}}}=a.$$

8. Find the quartic function of x which becomes 0 when $x=1, 3, -4$, or $2/3$, and has the value 48 when $x=0$.

9. In how many ways can m men and w women be seated in a row so that no two women are sitting side by side? $(w\leqq m+1)$

10. In the expansion of $(1+x)^n$, three consecutive coefficients are 45, 120, and 210. Find n and the numbers of the three terms.

II

11. Making use of the graph of x^2, trace the graph of

$$(a)\ -x^2,\quad (b)\ x^2+4,\quad (c)\ 4-x^2.$$

12. Solve $$\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{x-2}{x-1}+\frac{x-1}{x}.$$

13. If m and n are the roots of $3x^2-px+5=0$, find the equation whose roots are (*a*) m^2n and mn^2; (*b*) $2m-3n$ and $2n-3m$;

$$(c)\ \frac{1}{2m+n} \text{ and } \frac{1}{2n+m}.$$

14. Find the sum of $2n$ terms of the series whose rth term is $5-(-1)^r r$.

15. If $A>B$, write the proper sign between (*a*) $A+k$ and $B+k$, (*b*) kA and kB, when k is (i) a positive number, (ii) zero, (iii) a negative number.

16. Show that the largest rectangle which has a given perimeter is a square.

17. (*a*) Express as the product of two factors

$$a^3+b^3+c^3-3abc.$$

(*b*) Given that $a^3+b^3+c^3=3abc$, where a, b, and c are real numbers, show that either $a+b+c=0$, or $a=b=c$.

18. (*a*) A boy has 4 clear glass alleys and 8 coloured ones. The coloured ones are all different. How many combinations can he form, 4 at a time?

(*b*) Extend this to find a formula (simplified) for the number of combinations, n at a time, of $3n$ things of which n are alike of one kind and the rest all different.

19. If a_n denotes the coefficient of x^n in the expansion of

$$\frac{p+qx}{1+x+x^2},$$

prove that $a_n=a_{n-3}$.

20. A certain company makes personal loans on the following terms. In payment for a loan of \$300 made now, a borrower pays \$30 at the end of each month for the next 12 months. Find, correct to 0.1%, the rate of interest per annum paid by the borrower if interest is (*a*) simple, (*b*) compounded monthly.

III

21. Show that, in the harmonic progression $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, the sum of the terms from the $(k+1)$th to the $2k$th $(k>1)$, inclusive, is greater than $\frac{1}{2}$. What conclusion do you draw regarding the sum of n terms of the series as n increases indefinitely?

22. The nth term of a series is $2+n^2+2^n$. Find the sum of n terms.

23. Show that the quadratic function which takes the values p, q, r, when $x=a$, b, c, respectively, can be written in the form:

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}p+\frac{(x-a)(x-c)}{(b-a)(b-c)}q+\frac{(x-a)(x-b)}{(c-a)(c-b)}r.$$

By analogy, write an expression for a cubic function which takes the values p, q, r, s, when $x=a$, b, c, d, respectively.

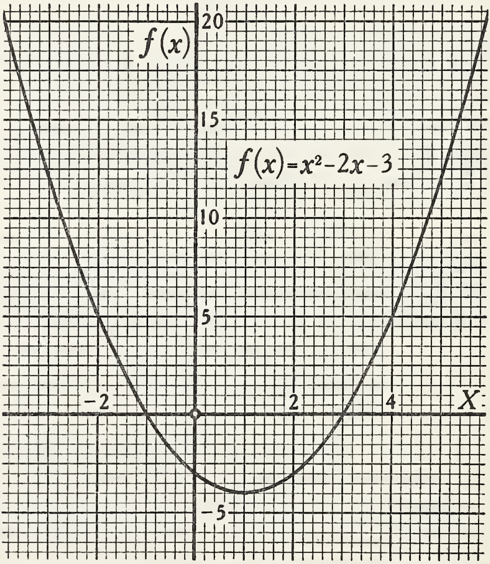
24. (*a*) Show that every polynomial with real coefficients is the product of real linear and quadratic factors.

(*b*) Express x^6+1 as the product of three real quadratic factors.

25. The accompanying figure shows the graph of the function $f(x)=x^2-2x-3$. From this graph determine the roots of the following equations:

(*a*) $x^2-2x-3=0$, (*b*) $x^2-2x-3=5$, (*c*) $f(x)=12$,
(*d*) $x^2-2x=0$, (*e*) $x^2-2x+1=0$.

What information does the graph give concerning the roots of $f(x)=-5$?



26. Solve $2x^4-x^3-11x^2-x+2=0$.

27. Show that, if all the letters denote positive numbers,

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$$

is not less than 9.

28. Given that $l^2+m^2+n^2=l'^2+m'^2+n'^2=1$, show that $1-(ll'+mm'+nn')^2=(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2$.

29. In how many ways can the letters of the word *alabaster* be arranged if no two *a*'s are ever side by side?

30. Find the first three terms and the general term in the expansion of $\dfrac{2+x+x^2}{(1-x)^2}$ (*a*) in ascending powers of x, (*b*) in descending powers of x. In each case specify the range of values of x for which the expansion represents the function.

IV

31. Simplify the product

$$(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)\ \cdots\ (2^{2^n}+1).$$

32. Solve $x^3-14x^2-84x+216=0$, given that its roots are in geometric progression.

33. If a weight of w pounds is tied to a string r feet long and swung in a circle with a speed of v feet per second, the centrifugal force F is given by the formula

$$F=\frac{wv^2}{32r}.$$

(*a*) If w and v are doubled and r is halved, what is the effect on F?

(*b*) If F and r are doubled while w remains constant, what is the effect on v?

(*c*) If w remains constant while v is doubled, what change must be made in r in order that F may be unchanged?

34. If x, a_1, a_2, y are in arithmetic progression, x, g_1, g_2, y are in geometric progression, and x, h_1, h_2, y are in harmonic progression, prove that

$$\frac{g_1g_2}{h_1h_2}=\frac{a_1+a_2}{h_1+h_2}.$$

35. Find the quadratic function of x which has a minimum value of 10 when $x=5$, and has the value 11 when $x=6$.

36. If $\left(x+\frac{1}{x}\right)^2=3$, prove that $x^3+\frac{1}{x^3}=0$.

37. A printed page is to be designed so as to contain 18 square inches of printed matter. The margins at the top and bottom are to be 2 inches each and those at the sides 1 inch each. Find the dimensions of the page which will require the least amount of paper.

38. If n points in a plane, no three of which are in the same straight line, are joined in all possible ways, find the number of intersections exclusive of the given points.

39. The applicants for 3 positions, A, B, and C, are 8 men and 5 women. Of these, 4 men and 2 women are over 40 years of age. Position A requires a person under 40 years of age. In how many ways can the positions be filled if at least one woman is appointed?

40. By comparing binomial expansions, prove that

$$\left(1+\frac{1}{m+1}\right)^{m+1}>\left(1+\frac{1}{m}\right)^m$$

where m is a positive integer.

V

41. Show that, if the formula $2n^2+5n$ gives the sum of n terms of a certain arithmetic series for two positive integral values of n, it gives the sum of n terms for all such values of n.

42. M varies as x when y is constant and as y when x is constant; N varies as x^2 when y is constant and as y^2 when x is constant. If $P=M+N$, find the definite relation connecting P with x and y, given: $P=10$ when $x=1$ and $y=2$, and $P=6$ when $x=-2$ and $y=1$.

43. What is the character of the graph of ax^2+c? What is its axis of symmetry? How is the graph altered if (i) a is changed while c remains constant, (ii) c is changed while a remains constant?

44. Show that there exists a quadratic function whose values are 28, 4, 12, 52, corresponding to $x=-4, 0, 4, 8$, respectively. Find the function.

45. Show that, for all real values of x, x^4+8x^2+3 is not less than $4x^3+8x$.

46. Solve the equation $x^3-3x^2-13x+15=0$, given that its roots are in arithmetic progression.

47. Find a formula for $\Sigma n^4=1^4+2^4+3^4+\cdots+n^4$.

48. On an ordinary checker-board of 64 squares, find the number of rectangles formed by the lines which bound the squares.

49. From a deck containing 52 different cards, 5 cards are drawn at random and then replaced, after which 4 cards are drawn. In how many ways could exactly 2 cards be common to the two drawings?

50. Find the coefficient of x^{3n-1} and of x^{3n+1} in the expansion of

$$\frac{2x-x^2}{8-x^3}.$$

VI

51. Find two numbers such that their sum diminished by their product is 1, and the square of their sum increased by the sum of their squares is 62.

52. If the roots of $x^3-abc\,x^2+(ab+bc+ca)x-a-b-c=0$ are l, m, n, show that the roots of

$$x^3-lmn\,x^2+(lm+mn+nl)x-l-m-n=0 \text{ are } a,\ b,\ c.$$

53. Find the equation of the locus of all points for which the equation

$$p^2(1-x^2)+2xyp+1-y^2=0$$

has equal roots for p. Construct the graph and discuss the character of the roots for points on each side of the locus.

54. Show that if $2a$ is the harmonic mean of the two rational numbers b and c, then the sum of the squares of the three numbers a, b, and c is the square of a rational number.

55. Find the sum of n terms of

(*a*) $9+99+999+9999+\cdots$,

(*b*) $4+44+444+4444+\cdots$.

56. Prove by mathematical induction that

$$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots \text{ to } n \text{ terms} =\frac{n(n+1)(n+2)(n+3)}{4}.$$

57. Express $3x^3-13x^2+14x+4$ as a polynomial in powers of $x-2$.

58. Prove that

$$_{2n}C_n=\frac{1\cdot3\cdot5\cdots(2n-1)}{n!}2^n.$$

59. In the Morse code, letters are represented by combinations of dots and dashes. How many letters would it be possible to represent by using not more than four of these marks for a letter?

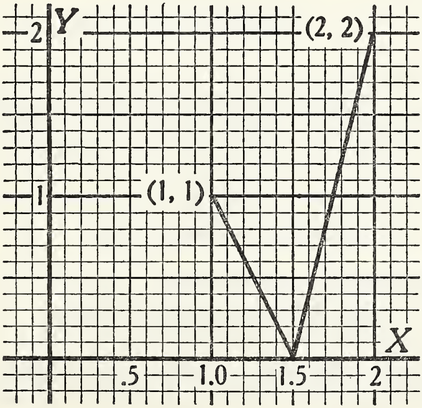
60. In the theory of the Michelson-Morley experiment, the theoretical effect was measured by

$$2(l_2+l_1)\left\{\frac{c}{c^2-v^2}-\frac{1}{\sqrt{c^2-v^2}}\right\},$$

where c is the velocity of light and v the earth's velocity relative to the ether. Show that, if $\frac{v}{c}$ is small, the expression is approximately given by $\frac{(l_2+l_1)v^2}{c^3}$.

VII

61. A function has the property $f(2x)=2f(x)$. A portion of its graph, from $x=1$ to $x=2$, is shown in the figure. Draw a graph of the function for values of x from $x=\frac{1}{8}$ to $x=8$.



62. (*a*) For an equation with real coefficients, state the possible numbers of real and imaginary, equal and unequal roots when the equation is (i) cubic, (ii) quartic, (iii) quintic.

(*b*) Give a simple example of a quartic equation with two equal real roots and two imaginary roots.

(*c*) Give a simple example of a quintic equation with one real root and two equal pairs of imaginary roots.

63. The natural numbers are divided into groups thus: 1; 2, 3; 4, 5, 6; 7, 8, 9, 10; . . . . (*a*) Find the sum of the numbers in the *m*th group. (*b*) Prove that the average value of the squares of the numbers in the *m*th group is

$$\frac{(3m^2+1)(m^2+2)}{12}.$$

64. Find the range of possible values of $\frac{x^2-1}{x^2-x+1}$ for real values of x.

65. Show that if a, b, and c are real numbers which are not all equal, then

$$bc(a-b)(a-c)+ca(b-c)(b-a)+ab(c-a)(c-b)$$

is positive.

66. Show that, if k is chosen so that the equation

$$ax^2+2hx+b+k(a'x^2+2h'x+b')=0$$

has equal roots, the common value of these roots is a root of

$$(ax+h)(h'x+b')-(a'x+h')(hx+b)=0.$$

67. Resolve into linear factors

$$6x^2-5y^2-12z^2+7xy+xz+19yz.$$

68. From an unlimited number of red, white, black, and blue balls, in how many ways can a selection of 10 be made?

69. Find a power of a binomial which yields an infinite series of which the initial terms are $1-3x+5x^2-\frac{55}{9}x^3+\cdots$.

70. A loan of \$5000 with compound interest at 5% is to be repaid, principal and interest, by annual payments, beginning one year after the loan is made. Each annual payment is to be \$500 except the last, which is to be less than \$500. Find the number of payments necessary and the amount of the last payment.

VIII

71. A number of spheres are arranged in a pile of which the bottom layer is rectangular, with m spheres along its length and n along its width. If the pile contains p complete layers, $(p \leqq n)$, find the total number of spheres.

72. If s_1, s_2, and s_3 are the sums of n terms of three arithmetic series whose first terms are 1 and whose common differences form a harmonic progression, prove that

$$n = \frac{2s_1s_3 - s_1s_2 - s_2s_3}{s_1 + s_3 - 2s_2}.$$

73. Show, by inspection, that the following equation is satisfied by 3 values of x and hence by every value:

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2.$$

74. (*a*) If $x^2+bx+c=0$ has equal roots, then

$$x^2+bx+c=(x-m)^2,$$

where m is the common value of the two roots. Find the relations among m, b, and c and hence obtain the condition satisfied by b and c when $x^2+bx+c=0$ has equal roots.

(*b*) Use a similar method to find the relations satisfied by b, c, and d when $x^3+bx^2+cx+d=0$ has all its three roots equal.

75. Resolve into partial fractions

$$\frac{16x+25}{12x^2-34x+10}.$$

76. By inspection of the following tables of values find, for each, a formula connecting *r* and *s*:

(*a*)

s	1	2	7/3	4	5	6.5
r	3.5	5	5.5	8	9.5	11.75

(*b*)

s	0	1	2	3	4	5	6
r	0	−1	0	3	8	15	24

77. Find all the solutions of the system

$$\begin{aligned} xy+\ yz+\ zx &= 0 \\ xy+2yz+3zx &= 0 \\ x^3+y^3+z^3 &= 12(y+z-x). \end{aligned}$$

78. Find the number of ways in which $2n$ beads of different colours can be made into two strings, $n+k$ beads being on one string and $n-k$ on the other.

79. Factor

$$\begin{aligned} &(kx-y+z)(x+ky-z)(x-y-kz) \\ &\qquad -(kx+y-z)(x-ky-z)(x-y+kz). \end{aligned}$$

80. Find the value of the infinite series

$$1+\frac{1}{4}+\frac{1\cdot 4}{4\cdot 8}+\frac{1\cdot 4\cdot 7}{4\cdot 8\cdot 12}+\cdots.$$

IX

81. Show that, if the sides of a right triangle are in arithmetic progression, then they are proportional to 3, 4, and 5.

82. If $f(x)$ has the property $f(mx)=mf(x)$ for all values of m, show that its graph is a straight line through the origin.

83. Determine b as a function of a in order that the sum of the squares of the roots of $x^3+ax^2+bx+c=0$ may be 4.

84. Show that, if a and b are real constants and x a real variable, the fraction

$$\frac{ax^2+bx-1}{x^2+x}$$

can take all real values if, and only if, $a-b \geqq 1$.

85. Find the sum of n terms of the series

$$1^2+2^2r+3^2r^2+4^2r^3+\cdots .$$

86. Solve the system

$$\begin{aligned} x^2+y^2-z^2 &= 0 \\ x+y+z &= 132 \\ xy &= 660. \end{aligned}$$

87. If a body of weight w, on a horizontal plane, is acted upon by a force P at an angle θ with the plane, the force of friction is $\mu(w-P \sin \theta)$, where μ is the coefficient of friction. When the body is on the point of moving, $P \cos \theta = \mu (w-P \sin \theta)$ whence

$$P=\frac{w\mu}{\cos \theta+\mu \sin \theta}.$$

If $\tan \theta=t$, this becomes

$$P=\frac{w\mu\sqrt{1+t^2}}{1+\mu t}.$$

Show that P is a minimum when $t=\mu$, that is, when θ is the angle of friction.

88. Find the number of ways in which $3n$ different books can be distributed among three persons so that the three numbers representing their shares may be in arithmetic progression. (Include the cases in which the numbers are $0, n, 2n$ and n, n, n.)

89. Determine a, b, c, d, and e in order that the nth term in the expansion of

$$\frac{a+bx+cx^2+dx^3+ex^4}{(1-x)^5}$$

in ascending powers of x may be n^4x^{n-1}.

90. Dominion of Canada $4\frac{1}{2}\%$ bonds, bearing half-yearly coupons and having 16 years to run, are sold at 112. Estimate the yield rate, correct to 0.1%.

X

91. Show that the numbers m^2+n^2, m^2-n^2, and $2mn$ represent the sides of a right-angled triangle. If m and n are integers, then the sides are represented by integers. Find all such triples of integers in which no integer exceeds 30.

92. Prove that a solution of any two of the following equations also satisfies the third.

$$\begin{aligned}(ad+be)x+(ae+bf)y+(af+bd)&=0\\(bd+ce)x+(be+cf)y+(bf+cd)&=0\\(cd+ae)x+(ce+af)y+(cf+ad)&=0.\end{aligned}$$

93. If p is greater than 1, prove that, for any real value of x other than $-p$ or p, the value of the function $\dfrac{x^2-2x+p^2}{x^2+2x+p^2}$ is greater than $\dfrac{p-1}{p+1}$ and less than $\dfrac{p+1}{p-1}$.

94. Find the quartic function of x which vanishes when $x=2$ and when $x=4$ and has the values of 24, 6, and -24 when $x=0$, 1, and 3, respectively.

95. Solve the system

$$\begin{aligned} x^3+y^3+z^3 &= 73 \\ x+y+z &= 7 \\ xyz &= 8. \end{aligned}$$

96. Prove, by mathematical induction, that the sum of the squares of the roots of

$$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0=0$$

is

$$\frac{a_{n-1}^2-2a_na_{n-2}}{a_n^2}.$$

97. Show that

$$ax^2+2hxy+by^2+2gx+2fy+c$$

can be expressed as the product of two linear factors if

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

(Hint: Complete the square of the terms containing x.)

98. From a rectangular solid, a corner is cut off by a plane which cuts the edges at distances x, $x+1$, and $x+2$ from the vertex of the corner. If the pyramid cut off has a volume of 3, find x to 3 figures.

99. Find the number of ways in which m like cards can be placed in $n+1$ numbered envelopes, if no restriction is made on the number to be placed in any envelope.

100. Prove the following relations connecting the binomial coefficients:

$$(a)\quad c_0-\frac{c_1}{2}+\frac{c_2}{3}-\cdots+(-1)^n\frac{c_n}{n+1}=\frac{1}{n+1};$$

$$(b)\quad \frac{c_1}{1}-\frac{c_2}{2}+\frac{c_3}{3}-\cdots+(-1)^{n-1}\frac{c_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

TABLE I

Amount of 1 at the end of *n* periods

$(1+i)^n$

n	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	n
1	1.005000	1.007500	1.010000	1.012500	1.015000	1.017500	1.020000	1.025000	1
2	1.010025	1.015056	1.020100	1.025156	1.030225	1.035306	1.040400	1.050625	2
3	1.015075	1.022669	1.030301	1.037971	1.045678	1.053424	1.061208	1.076891	3
4	1.020151	1.030339	1.040604	1.050945	1.061364	1.071859	1.082432	1.103813	4
5	1.025251	1.038067	1.051010	1.064082	1.077284	1.090617	1.104081	1.131408	5
6	1.030378	1.045852	1.061520	1.077383	1.093443	1.109702	1.126162	1.159693	6
7	1.035529	1.053696	1.072135	1.090850	1.109845	1.129122	1.148686	1.188686	7
8	1.040707	1.061599	1.082857	1.104486	1.126493	1.148882	1.171659	1.218403	8
9	1.045911	1.069561	1.093685	1.118292	1.143390	1.168987	1.195093	1.248863	9
10	1.051140	1.077583	1.104622	1.132271	1.160541	1.189444	1.218994	1.280085	10
11	1.056396	1.085664	1.115668	1.146424	1.177949	1.210260	1.243374	1.312087	11
12	1.061678	1.093807	1.126825	1.160755	1.195618	1.231439	1.268242	1.344889	12
13	1.066986	1.102010	1.138093	1.175264	1.213552	1.252990	1.293607	1.378511	13
14	1.072321	1.110276	1.149474	1.189955	1.231756	1.274917	1.319479	1.412974	14
15	1.077683	1.118603	1.160969	1.204829	1.250232	1.297228	1.345868	1.448298	15
16	1.083071	1.126992	1.172579	1.219890	1.268986	1.319929	1.372786	1.484506	16
17	1.088487	1.135445	1.184304	1.235138	1.288020	1.343028	1.400241	1.521618	17
18	1.093929	1.143960	1.196147	1.250577	1.307341	1.366531	1.428246	1.559659	18
19	1.099399	1.152540	1.208109	1.266210	1.326951	1.390445	1.456811	1.598650	19
20	1.104896	1.161184	1.220190	1.282037	1.346855	1.414778	1.485947	1.638616	20
21	1.110420	1.169893	1.232392	1.298063	1.367058	1.439537	1.515666	1.679582	21
22	1.115972	1.178667	1.244716	1.314288	1.387564	1.464729	1.545980	1.721571	22
23	1.121552	1.187507	1.257163	1.330717	1.408377	1.490361	1.576899	1.764611	23
24	1.127160	1.196414	1.269735	1.347351	1.429503	1.516443	1.608437	1.808726	24
25	1.132796	1.205387	1.282432	1.364193	1.450945	1.542981	1.640606	1.853944	25
26	1.138460	1.214427	1.295256	1.381245	1.472710	1.569983	1.673418	1.900293	26
27	1.144152	1.223535	1.308209	1.398511	1.494800	1.597457	1.706886	1.947800	27
28	1.149873	1.232712	1.321291	1.415992	1.517222	1.625413	1.741024	1.996495	28
29	1.155622	1.241957	1.334504	1.433692	1.539981	1.653858	1.775845	2.046407	29
30	1.161400	1.251272	1.347849	1.451613	1.563080	1.682800	1.811362	2.097568	30
31	1.167207	1.260656	1.361327	1.469759	1.586526	1.712249	1.847589	2.150007	31
32	1.173043	1.270111	1.374941	1.488131	1.610324	1.742213	1.884541	2.203757	32
33	1.178908	1.279637	1.388690	1.506732	1.634479	1.772702	1.922231	2.258851	33
34	1.184803	1.289234	1.402577	1.525566	1.658996	1.803725	1.960676	2.315322	34
35	1.190727	1.298904	1.416603	1.544636	1.683881	1.835290	1.999890	2.373205	35
36	1.196681	1.308645	1.430769	1.563944	1.709140	1.867407	2.039887	2.432535	36
37	1.202664	1.318460	1.445076	1.583493	1.734777	1.900087	2.080685	2.493349	37
38	1.208677	1.328349	1.459527	1.603287	1.760798	1.933338	2.122299	2.555682	38
39	1.214721	1.338311	1.474123	1.623328	1.787210	1.967172	2.164745	2.619574	39
40	1.220794	1.348349	1.488864	1.643619	1.814018	2.001597	2.208040	2.685064	40

TABLE I

Amount of 1 at the end of *n* periods

$$(1+i)^n$$

n	3%	$3\frac{1}{2}$%	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%	7%	n
1	1.030000	1.035000	1.040000	1.045000	1.050000	1.055000	1.060000	1.070000	1
2	1.060900	1.071225	1.081600	1.092025	1.102500	1.113025	1.123600	1.144900	2
3	1.092727	1.108718	1.124864	1.141166	1.157625	1.174241	1.191016	1.255043	3
4	1.125509	1.147523	1.169859	1.192519	1.215506	1.238825	1.262477	1.310796	4
5	1.159274	1.187686	1.216653	1.246182	1.276282	1.306960	1.338226	1.402552	5
6	1.194052	1.229255	1.265319	1.302260	1.340096	1.378843	1.418519	1.500730	6
7	1.229874	1.272279	1.315932	1.360862	1.407100	1.454679	1.503630	1.605781	7
8	1.266770	1.316809	1.368569	1.422101	1.477455	1.534687	1.593848	1.718186	8
9	1.304773	1.362897	1.423312	1.486095	1.551328	1.619094	1.689479	1.838459	9
10	1.343916	1.410599	1.480244	1.552969	1.628895	1.708144	1.790848	1.967151	10
11	1.384234	1.459970	1.539454	1.622853	1.710339	1.802092	1.898299	2.104852	11
12	1.425761	1.511069	1.601032	1.695881	1.795856	1.901207	2.012196	2.252192	12
13	1.468534	1.563956	1.665074	1.772196	1.885649	2.005774	2.132928	2.409845	13
14	1.512590	1.618695	1.731676	1.851945	1.979932	2.116091	2.260904	2.578534	14
15	1.557967	1.675349	1.800944	1.935282	2.078928	2.232476	2.396558	2.759032	15
16	1.604706	1.733986	1.872981	2.022370	2.182875	2.355263	2.540352	2.952164	16
17	1.652848	1.794676	1.947900	2.113377	2.292018	2.484802	2.692773	3.158815	17
18	1.702433	1.857489	2.025817	2.208479	2.406619	2.621466	2.854339	3.379932	18
19	1.753506	1.922501	2.106849	2.307860	2.526950	2.765647	3.025599	3.616528	19
20	1.806111	1.989789	2.191123	2.411714	2.653298	2.917757	3.207135	3.869684	20
21	1.860295	2.059431	2.278768	2.520241	2.785963	3.078234	3.399564	4.140562	21
22	1.916103	2.131512	2.369919	2.633652	2.925261	3.247537	3.603537	4.430402	22
23	1.973587	2.206114	2.464716	2.752166	3.071524	3.426152	3.819750	4.740530	23
24	2.032794	2.283328	2.563304	2.876014	3.225100	3.614590	4.048935	5.072367	24
25	2.093778	2.363245	2.665836	3.005434	3.386355	3.813392	4.291871	5.427433	25
26	2.156591	2.445959	2.772470	3.140679	3.555673	4.023129	4.549383	5.807353	26
27	2.221289	2.531567	2.883369	3.282010	3.733456	4.244401	4.822346	6.213868	27
28	2.287928	2.620172	2.998703	3.429700	3.920129	4.477843	5.111687	6.648838	28
29	2.356566	2.711878	3.118651	3.584036	4.116136	4.724124	5.418388	7.114257	29
30	2.427262	2.806794	3.243398	3.745318	4.321942	4.983951	5.743491	7.612255	30
31	2.500080	2.905031	3.373133	3.913857	4.538039	5.258069	6.088101	8.145113	31
32	2.575083	3.006708	3.508059	4.089981	4.764941	5.547262	6.453387	8.715271	32
33	2.652335	3.111942	3.648381	4.274030	5.003189	5.852362	6.840590	9.325340	33
34	2.731905	3.220860	3.794316	4.466362	5.253348	6.174242	7.251025	9.978114	34
35	2.813862	3.333590	3.946089	4.667348	5.516015	6.513825	7.686087	10.676581	35
36	2.898278	3.450266	4.103933	4.877378	5.791816	6.872085	8.147252	11.423942	36
37	2.985227	3.571025	4.268090	5.096860	6.081407	7.250050	8.636087	12.223618	37
38	3.074783	3.696011	4.438813	5.326219	6.385477	7.648803	9.154252	13.079271	38
39	3.167027	3.825372	4.616366	5.565899	6.704751	8.069487	9.703507	13.994820	39
40	3.262038	3.959260	4.801021	5.816365	7.039989	8.513309	10.285718	14.974458	40

TABLE II

Present value of 1 due at the end of *n* periods

$1/(1+i)^n$

n	½%	¾%	1%	1¼%	1½%	1¾%	2%	2½%	*n*
1	0.995025	0.992556	0.990099	0.987654	0.985222	0.982801	0.980392	0.975610	1
2	0.990075	0.985167	0.980296	0.975461	0.970662	0.965898	0.961169	0.951814	2
3	0.985149	0.977833	0.970590	0.963418	0.956317	0.949285	0.942322	0.928599	3
4	0.980248	0.970554	0.960980	0.951524	0.942184	0.932959	0.923845	0.905951	4
5	0.975371	0.963329	0.951466	0.939777	0.928260	0.916913	0.905731	0.883854	5
6	0.970518	0.956158	0.942045	0.928175	0.914542	0.901143	0.887971	0.862297	6
7	0.965690	0.949040	0.932718	0.916716	0.901027	0.885644	0.870560	0.841265	7
8	0.960885	0.941975	0.923483	0.905398	0.887711	0.870412	0.853490	0.820747	8
9	0.956105	0.934963	0.914340	0.894221	0.874592	0.855441	0.836755	0.800728	9
10	0.951348	0.928003	0.905287	0.883181	0.861667	0.840729	0.820348	0.781198	10
11	0.946615	0.921095	0.896324	0.872277	0.848933	0.826269	0.804263	0.762145	11
12	0.941905	0.914238	0.887449	0.861509	0.836387	0.812058	0.788493	0.743556	12
13	0.937219	0.907432	0.878663	0.850873	0.824027	0.798091	0.773033	0.725420	13
14	0.932556	0.900677	0.869963	0.840368	0.811849	0.784365	0.757875	0.707727	14
15	0.927917	0.893973	0.861349	0.829993	0.799851	0.770875	0.743015	0.690466	15
16	0.923300	0.887318	0.852821	0.819746	0.788031	0.757616	0.728446	0.673625	16
17	0.918707	0.880712	0.844377	0.809626	0.776385	0.744586	0.714163	0.657195	17
18	0.914136	0.874156	0.836017	0.799631	0.764912	0.731780	0.700159	0.641166	18
19	0.909588	0.867649	0.827740	0.789759	0.753607	0.719194	0.686431	0.625528	19
20	0.905063	0.861190	0.819544	0.780009	0.742470	0.706825	0.672971	0.610271	20
21	0.900560	0.854779	0.811430	0.770379	0.731498	0.694668	0.659776	0.595386	21
22	0.896080	0.848416	0.803396	0.760868	0.720688	0.682720	0.646839	0.580865	22
23	0.891622	0.842100	0.795442	0.751475	0.710037	0.670978	0.634156	0.566697	23
24	0.887186	0.835831	0.787566	0.742197	0.699544	0.659438	0.621721	0.552875	24
25	0.882772	0.829609	0.779768	0.733034	0.689206	0.648096	0.609531	0.539391	25
26	0.878380	0.823434	0.772048	0.723984	0.679021	0.636950	0.597579	0.526235	26
27	0.874010	0.817304	0.764404	0.715046	0.668986	0.625995	0.585862	0.513400	27
28	0.869662	0.811220	0.756836	0.706219	0.659099	0.615228	0.574375	0.500878	28
29	0.865335	0.805181	0.749342	0.697500	0.649359	0.604647	0.563112	0.488661	29
30	0.861030	0.799187	0.741923	0.688889	0.639762	0.594248	0.552071	0.476743	30
31	0.856746	0.793238	0.734577	0.680384	0.630308	0.584027	0.541246	0.465115	31
32	0.852484	0.787333	0.727304	0.671984	0.620993	0.573982	0.530633	0.453771	32
33	0.848242	0.781472	0.720103	0.663688	0.611816	0.564111	0.520229	0.442703	33
34	0.844022	0.775654	0.712973	0.655494	0.602774	0.554408	0.510028	0.431905	34
35	0.839823	0.769880	0.705914	0.647402	0.593866	0.544873	0.500028	0.421371	35
36	0.835645	0.764149	0.698925	0.639409	0.585090	0.535502	0.490223	0.411094	36
37	0.831487	0.758461	0.692005	0.631515	0.576443	0.526292	0.480611	0.401067	37
38	0.827351	0.752814	0.685153	0.623719	0.567924	0.517240	0.471187	0.391285	38
39	0.823235	0.747210	0.678370	0.616018	0.559531	0.508344	0.461948	0.381741	39
40	0.819139	0.741648	0.671653	0.608413	0.551262	0.499601	0.452890	0.372431	40

TABLE II

Present value of 1 due at the end of *n* periods

$1/(1+i)^n$

n	3%	3½%	4%	4½%	5%	5½%	6%	7%	*n*
1	0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.934579	1
2	0.942596	0.933511	0.924556	0.915730	0.907029	0.898452	0.889996	0.873439	2
3	0.915142	0.901943	0.888996	0.876297	0.863838	0.851614	0.839619	0.816298	3
4	0.888487	0.871442	0.854804	0.838561	0.822702	0.807217	0.792094	0.762895	4
5	0.862609	0.841973	0.821927	0.802451	0.783526	0.765134	0.747258	0.712986	5
6	0.837484	0.813501	0.790315	0.767896	0.746215	0.725246	0.704961	0.666342	6
7	0.813092	0.785991	0.759918	0.734828	0.710681	0.687437	0.665057	0.622750	7
8	0.789409	0.759412	0.730690	0.703185	0.676839	0.651599	0.627412	0.582009	8
9	0.766417	0.733731	0.702587	0.672904	0.644609	0.617629	0.591898	0.543934	9
10	0.744094	0.708919	0.675564	0.643928	0.613913	0.585431	0.558395	0.508349	10
11	0.722421	0.684946	0.649581	0.616199	0.584679	0.554910	0.526788	0.475093	11
12	0.701380	0.661783	0.624597	0.589664	0.556837	0.525982	0.496969	0.444012	12
13	0.680951	0.639404	0.600574	0.564272	0.530321	0.498561	0.468839	0.414964	13
14	0.661118	0.617782	0.577475	0.539973	0.505068	0.472569	0.442301	0.387817	14
15	0.641862	0.596891	0.555264	0.516720	0.481017	0.447933	0.417265	0.362446	15
16	0.623167	0.576706	0.533908	0.494469	0.458112	0.424581	0.393646	0.338735	16
17	0.605016	0.557204	0.513373	0.473176	0.436297	0.402447	0.371364	0.316574	17
18	0.587395	0.538361	0.493628	0.452800	0.415521	0.381466	0.350344	0.295864	18
19	0.570286	0.520156	0.474642	0.433302	0.395734	0.361579	0.330513	0.276508	19
20	0.553676	0.502566	0.456387	0.414643	0.376889	0.342729	0.311805	0.258419	20
21	0.537549	0.485571	0.438834	0.396787	0.358942	0.324862	0.294155	0.241513	21
22	0.521892	0.469151	0.421955	0.379701	0.341850	0.307926	0.277505	0.225713	22
23	0.506692	0.453286	0.405726	0.363350	0.325571	0.291873	0.261797	0.210947	23
24	0.491934	0.437957	0.390121	0.347703	0.310068	0.276657	0.246979	0.197147	24
25	0.477606	0.423147	0.375117	0.332731	0.295303	0.262234	0.232999	0.184249	25
26	0.463695	0.408838	0.360689	0.318402	0.281241	0.248563	0.219810	0.172195	26
27	0.450189	0.395012	0.346817	0.304691	0.267848	0.235604	0.207368	0.160930	27
28	0.437077	0.381654	0.333477	0.291571	0.255094	0.223322	0.195630	0.150402	28
29	0.424346	0.368748	0.320651	0.279015	0.242946	0.211679	0.184557	0.140563	29
30	0.411987	0.356278	0.308319	0.267000	0.231377	0.200644	0.174110	0.131367	30
31	0.399987	0.344230	0.296460	0.255502	0.220359	0.190184	0.164255	0.122773	31
32	0.388337	0.332590	0.285058	0.244500	0.209866	0.180269	0.154957	0.114741	32
33	0.377026	0.321343	0.274094	0.233971	0.199873	0.170871	0.146186	0.107235	33
34	0.366045	0.310476	0.263552	0.223896	0.190355	0.161963	0.137912	0.100219	34
35	0.355383	0.299977	0.253415	0.214254	0.181290	0.153520	0.130105	0.093663	35
36	0.345032	0.289833	0.243669	0.205028	0.172657	0.145516	0.122741	0.087535	36
37	0.334983	0.280032	0.234297	0.196199	0.164436	0.137930	0.115793	0.081809	37
38	0.325226	0.270562	0.225285	0.187750	0.156605	0.130739	0.109239	0.076457	38
39	0.315754	0.261412	0.216621	0.179665	0.149148	0.123924	0.103056	0.071455	39
40	0.306557	0.252572	0.208289	0.171929	0.142046	0.117463	0.097222	0.066780	40

REVIEW OF BASIC SKILLS

SET No. 1: SUBSTITUTION

If $x=6$, $y=-2$, $z=4$, $w=0$, evaluate the following:

1. $x^2+y^2+z^2+w^2$.

2. $x^2-y^2-z^2$.

3. $xy+yz+zw+wx$.

4. x^3-y^3.

5. $2x^2-5x+3$.

6. $4y^2+2y-7$.

7. $x^3+y^3-z^3+3xyz$.

8. $2z^3-3z^2+z-9$.

9. y^3-5y^2+7.

10. $(x+y)/(x-y)$.

11. $1/x+1/y+w/z$.

12. $w(x^3+17y^3-35z^3)$.

13. $(x^2-4y^2)/(3x-y)$.

14. xy^2-zx^2.

15. Find the value of $(-b+\sqrt{b^2-4ac})/2a$ when (i) $a=1$, $b=-2$, $c=-3$; (ii) $a=-2$, $b=8$, $c=-1$; (iii) $a=3$, $b=6$, $c=3$.

16. Find the value of x^2-3x+7 when (*a*) $x=b$; (*b*) $x=c+d$; (*c*) $x=1/a$.

17. Find the value of $(m-4)/(m+4)$ when (*a*) $m=-x$; (*b*) $m=1/x$; (*c*) $m=x+y$.

18. Find the value of x^2+y^2 when $x=a+b$ and $y=a-b$.

19. Find the value of $x/(1+x^2)$ when (*a*) $x=-1$; (*b*) $x=1/y$; (*c*) $x=y+1$.

20. Find the value of x^2-a^2 when (*a*) $x=a$; (*b*) $x=-a$; (*c*) $x=a+1$.

SET No. 2: ADDITION AND SUBTRACTION

Perform the indicated additions and subtractions in the following:

1. $(2a-3b+4c)+(5a-6b+8c)$.
2. $(3x-2y)-(4y-5x)$.
3. $(5x^2-2xy+3y^2)-(3x^2+5xy-6y^2)$.
4. $(7z^3-4z^2+3z-8)-(-2z^3+4z^2-5z+5)$.
5. $(x^2+4ax+c)+(3x^2+2ax-3c)-(x^2-ax-2c)$.
6. Subtract $3x^2-4xy-6y^2+6x$ from $5x^2-xy+y^2+6x$.
7. From the sum of $5a-7b-4c$ and $8b-4a+2c$, subtract the sum of $3c-8b+2a$ and $6b+7c-a$.
8. $(k^3-3km-5km^2+m^3)+(3k^3+4km-8m^3)$
$-(4k^3+km-5km^2-9m^3)$.

SET No. 3: MULTIPLICATION AND DIVISION

COMMON EXPANSIONS

$$(a\pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a\pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$(a-b)(a+b) = a^2 - b^2$$

Multiply:

1. $-5x^2y^2$, $3xy^2$.
2. $-2ab$, $-9a^2b^3$.
3. $3x-1$, $2x+7$.
4. $7m^2-5n$, $9m^2+3n$.
5. $a+2b$, $a^2-2ab+4b^2$.
6. $8x+3y$, $8x-3y$.

7. $a+2b-c,\ a+2b+c$.

8. $3m^2-2n^2,\ 3m^2+2n^2$.

9. $a+b,\ a^2-ab+b^2$.

10. $z^2+z+1,\ z^2+z-1$.

Expand the following:

11. $(4a+5b)^2$.

12. $(2xy-3c)^2$.

13. $(a+b)^3$.

14. $(x+y+z)^2$.

15. $(x-y)^3$.

16. $(1-2x)^3$.

17. $(2x-3y+4z)^2$.

18. $(3m+n)^3$.

19. $(3xy^2-2z)^2$.

20. $(5mn+3/mn)^2$.

21. $(a+1/a)^2$.

22. $(x-1/x)^3$.

Divide:

23. $-7s^3t$ by $2st$.

24. $-32a^5b^4c^3$ by $-8a^3b^3c^3$.

25. $a^3b^3-a^2b^4-5a^2b^2$ by $-a^2b^2$.

26. $6x^3-7x^2-50$ by $2x-5$.

27. $3x^3-5x^2+3x-1$ by $x-1$.

28. $x^4+x^2y^2+y^4$ by x^2-xy+y^2.

29. $3x^4-5x^3y-12x^2y^2-xy^3+3y^4$ by $3x^2+xy-y^2$.

30. $14a^4-a^3-29a^2+12$ by $7a^2+3a-6$.

Simplify:

31. $(3x-4y)^2-3(2x+y)(2x-y)-(2x+3y)^2$.

32. $4(2x+5)^2-5(x-3)^2-13(x+5)(x-5)$.

33. $(k+m-n)^2+(k-m+n)^2-2(k-m-n)^2$.

34. $(x^2-5x+8)(4x^2+3x+3)-2(2x^2+5)(x^2+6x-5)$.

SET No. 4: FACTORING

DEFINITION: *To factor a polynomial means to express it as the product of polynomials of the same or lower degrees.*

SUMMARY OF FACTORING

Type, suggested by:	Identification	Method of Factoring
1. *Common factor*	*A factor is common to every term.*	(*Com. factor*) ( )
2. *Trinomial*	*Three terms.*	*Intelligent guessing.*
3. *Difference of squares*	$(\quad)^2-(\quad)^2$.	*Formula*: $(a+b)(a-b)$.
4. *Grouping*	*Four or more terms.*	*Group to get a common factor, difference of squares, or a trinomial.*
5. *Difference or sum of cubes*	$(\quad)^3\pm(\quad)^3$.	*Formula*: $(a\pm b)(a^2\mp ab+b^2)$.
6. *Incomplete squares*	*Fourth powers usually involved; perfect square except for defective product term.*	*Complete the square and subtract added part, thus getting* $(\quad)^2-(\quad)^2$.
7. *Factor theorem*	*Any polynomial.*	*If the polynomial*$=0$ *when* $x=a$, *then* $x-a$ *is a factor.*

Factor:

1. x^2-5x+6.
2. $4x^2-y^2$.
3. $8a^3+b^3$.
4. $p(x-m)-(x-m)$.

5. $x^3-4x^2-3x+12$.

Solution: $x^2(x-4)-3(x-4)$

$=(x-4)(x^2-3)$

$=(x-4)(x+\sqrt{3})(x-\sqrt{3})$.

6. $3x^4z+24xz^4$.

Solution: $3xz(x^3+8z^3)$

$=3xz(x+2z)(x^2-2xz+4z^2)$.

7. $m^3+5m^2-9m-45$.

8. $54a^4+128ab^3$.

9. x^4-4x^2.

10. $4x^2y^2z^3-8x^3y^2z^2+12x^2y^2z^3$.

11. $27x^4-64xy^3$.

12. $4x^2+4x-15$.

13. $x^2-2xy+y^2-z^2$.

Solution: $(x^2-2xy+y^2)-z^2$

$=(x-y)^2-z^2$

$=(x-y+z)(x-y-z)$.

14. x^4-x^2-12.

Solution: $(x^2-4)(x^2+3)$

$=(x-2)(x+2)(x^2+3)$.

15. $1+2ab-a^2-b^2$.

16. m^4-5m^2+4.

17. a^2-b^2+a+b.

18. $2a^2-22a-52$.

19. $mx^2+5mx-nx^2-5nx$.

20. $6x^2+5x-4$.

21. x^6-y^6.

22. $21x^2+10x-24$.

23. $a^{12}-b^{12}$.

24. $12a^2-41a+24$.

25. $x^4+x^2y^2+y^4$.

Solution: $(x^4+2x^2y^2+y^4)-x^2y^2$

$=(x^2+y^2)^2-(xy)^2$

$=(x^2+y^2+xy)(x^2+y^2-xy)$.

26. $4x^4-5x^2+1$.

27. $4m^4+1$.

28. $x^3-8y^3-5x^2y+10xy^2$.

29. $27ab^2-147ac^2$.

30. $6y^2-yz-15z^2$.

31. $9x^2-4y^2-16z^2+16yz$.

32. $18x^2y^2-57xy^2+45y^2$.

33. $2px+15qy-3qx-10py$.

34. $(4a+3)^2-(4a+3)(a-2)$.

35. $7a^4x^4+28a^4y^4$.

36. $a^4-15a^2b^2+9b^4$.

37. $m^4+n^4-18m^2n^2$.

38. Show that $x-1$ is a factor of $3x^3-5x^2+x+1$.

39. Show that $x+3$ is a factor of $4x^3+12x^2-x-3$.

40. Show that $2a-3$ is a factor of $6a^3+25a^2-31a-30$, and find the other two factors.

SET No. 5: FRACTIONS

BASIC PRINCIPLE

The terms of a fraction can be multiplied or divided by the same number ($\neq 0$) without changing the value of the fraction.

Reduce to lowest terms:

1. $\dfrac{m^2-mx}{m^2-x^2}$.

2. $\dfrac{x^2-7x+10}{x^2-5x+6}$.

3. $\dfrac{x^2-y^2+x-y}{x^2-2xy+y^2}$.

Solution:

$$\frac{(x+y)(x-y)+(x-y)}{(x-y)^2}=\frac{\overset{1}{\cancel{(x-y)}}\,(x+y+1)}{\underset{x-y}{\cancel{(x-y)^2}}}=\frac{x+y+1}{x-y}.$$

4. $\dfrac{(a-b)^2}{mb-ma}$.

5. $\dfrac{a^5-8a^2}{a^3-2a^2}$.

6. $\dfrac{20x^3y^2(25x^2-9)}{15x^2y^3(25x^2-30x+9)}$.

7. $\dfrac{3x-4}{27x^3-64}$.

8. $\dfrac{x^2-y^2-x-y}{x^2+2xy+y^2}$.

9. $\dfrac{x^4+x^2y^2+y^4}{x^6-y^6}$.

Simplify:

10. $1/x+1/y$.

11. $\dfrac{1/x+1/y}{1/x-1/y}$.

12. $\dfrac{1}{x-y}-\dfrac{1}{x+y}$.

13. $\dfrac{1}{x-y}\cdot\dfrac{1}{x+y}$.

14. $\dfrac{2a+3}{6a}-\dfrac{3a^2+1}{12a^2}-\dfrac{3a^3-4a}{24a^3}$.

15. $\dfrac{a^2-b^2}{a^2-4b^2}\div\dfrac{a+b}{a-2b}$.

16. $\dfrac{a}{a-b}-\dfrac{b}{a}-\dfrac{a^2b+ab^2}{a^3-ab^2}$.

Solution: $\dfrac{a}{a-b}-\dfrac{b}{a}-\dfrac{\overset{1}{\cancel{a}}b\,\overset{1}{(\cancel{a+b})}}{\underset{1}{\cancel{a}}\underset{1}{(\cancel{a+b})}\,(a-b)}$

$$=\frac{a^2-b(a-b)-ba}{a(a-b)}$$

$$=\frac{a^2-2ab+b^2}{a(a-b)}=\frac{\overset{a-b}{\cancel{(a-b)^2}}}{a\,\underset{1}{(\cancel{a-b})}}=\frac{a-b}{a}.$$

17. $\dfrac{m}{m^3-m}+\dfrac{m^2+5m+4}{m^2+3m-4}-\dfrac{2}{m+1}$.

18. $\dfrac{x^3-y^3}{x^3+y^3}\cdot\dfrac{x^2-y^2}{x^2+xy+y^2}$.

19. $\dfrac{x-3}{x-2}\cdot\dfrac{x^2-9x-22}{x^2-10x+21}\div\dfrac{x-11}{x^2-49}.$

20. $\dfrac{4x^2-9y^2}{2y^2-5xy-3x^2}\cdot\dfrac{3x^2+11xy-4y^2}{7xy-6x^2-2y^2}\div\dfrac{2x^2+11xy+12y^2}{4y^2-x^2}.$

21. $\dfrac{(2a+3)^2}{2a^2+13a+15}+\dfrac{a^2+a-30}{a^2-10a+25}-\dfrac{3a^2+4a}{a^2-25}.$

22. $\dfrac{x^2-y^2-x-y}{x^2y-xy^2-xy}\cdot\dfrac{x^3-2x^2-5x+10}{x^4-25}\div\dfrac{x^3-8}{yx^3+5xy}.$

23. $\dfrac{5x^2-7x-6}{30x^2+53x+21}+\dfrac{2}{7-6x}+\dfrac{24x}{36x^2-49}.$

24. $\dfrac{2}{m^2-1}-\dfrac{5}{3(m^2+m+1)}-\dfrac{2m^2-3m-8}{3(1-m^3)(m+1)}.$

25. $\dfrac{x+y}{(y-z)(z-x)}+\dfrac{y+z}{(z-x)(x-y)}+\dfrac{z+x}{(x-y)(y-z)}.$

26. $\dfrac{\dfrac{1}{x-y}+\dfrac{1}{x+y}}{\dfrac{1}{x-y}-\dfrac{1}{x+y}}.$ 27. $\dfrac{x^2-xy+y^2-\dfrac{y^3}{x+y}}{1-\dfrac{y}{x+y}}.$

SET No. 6: SURDS

Express as entire surds:

1. $2\sqrt{2}$. SOLUTION: $2\sqrt{2}=\sqrt{4}\cdot\sqrt{2}=\sqrt{8}$.

2. $3\sqrt{3}$. 3. $2\sqrt{5}$. 4. $6\sqrt{3}$.

5. $5\sqrt{6}$. 6. $\frac{3}{4}\sqrt{32}$.

Express as mixed surds:

7. $\sqrt{18}$. SOLUTION: $\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}$.

8. $\sqrt{8}$. 9. $\sqrt{75}$. 10. $\sqrt{125}$.

11. $\sqrt{a^2b}$. 12. $\sqrt[3]{16}$. 13. $\sqrt{108}$.

14. $\sqrt[3]{81}$. 15. $\sqrt{x^3y^3}$.

Simplify:

16. $8\sqrt{3}+2\sqrt{3}$.

17. $25\sqrt{6}+10\sqrt{6}-14\sqrt{6}$.

18. $5\sqrt{5}-7\sqrt{5}-12\sqrt{5}$.

19. $3\sqrt{7}+2\sqrt{10}-\sqrt{7}-10\sqrt{10}$.

20. $4\sqrt[3]{2}+6\sqrt[3]{2}-\sqrt[3]{2}$.

21. $17\sqrt[4]{5}-3\sqrt[4]{5}-10\sqrt[4]{5}$.

22. $\sqrt{8}+\sqrt{2}$.

23. $2\sqrt{27}+2\sqrt{75}-\sqrt{300}$.

24. $2\sqrt[3]{16}-4\sqrt[3]{2}$.

25. $\sqrt[3]{24}-3\sqrt[3]{81}+5\sqrt[3]{3}$.

26. $6\sqrt{50}+2\sqrt{18}-5\sqrt{8}$.

27. $\sqrt{125}+6\sqrt{5}-3\sqrt{20}$.

Multiply:

28. $(\sqrt{3}+\sqrt{2})(4\sqrt{3}-3\sqrt{2})$.

SOLUTION: $(\sqrt{3}+\sqrt{2})(4\sqrt{3}-3\sqrt{2})=12-3\sqrt{6}+4\sqrt{6}-6=6+\sqrt{6}$.

29. $(3\sqrt{3}+\sqrt{5})(2\sqrt{3}-\sqrt{5})$.

30. $(6+3\sqrt{5})(6-3\sqrt{5})$.

31. $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$.

32. $(8-4\sqrt{6})(10+3\sqrt{6})$.

33. $(4\sqrt{3}+\sqrt{2})(\sqrt{2}-4\sqrt{3})$.

34. $(4\sqrt{10}+\sqrt{6})^2-(4\sqrt{10}-\sqrt{6})^2$.

35. $(3\sqrt{5}-4)^2-(3\sqrt{5}-4)(3\sqrt{5}+4)$.

36. $(2\sqrt{3}-3\sqrt{2}-1)^2$.

Rationalize the denominator of each of the following:

37. $1/\sqrt{5}$. SOLUTION: $\dfrac{1}{\sqrt{5}}=\dfrac{1}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{\sqrt{5}}{5}=\dfrac{1}{5}\sqrt{5}$.

38. $4/(\sqrt{5}+\sqrt{3})$.

SOLUTION:

$$\frac{4}{\sqrt{5}+\sqrt{3}}=\frac{4(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}=\frac{4(\sqrt{5}-\sqrt{3})}{5-3}=2(\sqrt{5}-\sqrt{3}).$$

39. $1/\sqrt{2}$.
40. $3/\sqrt{3}$.
41. $-2/(\sqrt{3}-\sqrt{2})$.
42. $7/(3-2\sqrt{2})$.
43. $3(\sqrt{7}+\sqrt{5})/(\sqrt{7}-\sqrt{5})$.
44. $25/(9+2\sqrt{14})$.
45. $(2\sqrt{5}+3\sqrt{6})/(2\sqrt{5}-3\sqrt{6})$.
46. $(3\sqrt{11}-\sqrt{3})/(\sqrt{11}+\sqrt{3})$.

SET No. 7: EQUATIONS AND INEQUALITIES

Solve for x:

1. $x+11=2$.
2. $x+a=b$.
3. $x-8=-5$.
4. $x-a=-b$.
5. $7x=6$.
6. $ax=b$.
7. $x/3=-4$.
8. $x/a=b$.
9. $x\sqrt{3}=2$.
10. $8x=5x+.12$.
11. $ax=bx+c$.
12. $x\sqrt{3}=x\sqrt{2}-5$.
13. $kx+n=m$.
14. $5+10x=x+9.5$.
15. $x/4+x/5=9$.
16. $\dfrac{5}{2-x}=\dfrac{3}{x+4}$.
17. $\dfrac{3x-2}{2x-3}-\dfrac{x+17}{x+10}=\dfrac{1}{2}$.
18. $\dfrac{3+x}{3-x}-\dfrac{2+x}{2-x}=1-\dfrac{x+1}{x-1}$.
19. $\dfrac{3}{5}(2x-5)+\dfrac{x}{4}=\dfrac{1}{2}\left(\dfrac{3}{2}-x\right)-\dfrac{48}{5}$.
20. $\dfrac{2}{3}(5-2x)-\dfrac{1}{4}\left(3x-\dfrac{2}{3}\right)=2-\dfrac{x}{3}$
21. $\dfrac{2}{3}(x-1)-\dfrac{2}{5}(4x-5)=3-\dfrac{7x+5}{10}$.
22. $x+\sqrt{2x+1}=7$.

SOLUTION: If $\sqrt{2x+1}=7-x$

$\therefore\ 2x+1 = 49-14x+x^2$

$\therefore\ x^2-16x+48 = 0$

$\therefore\ (x-12)(x-4) = 0$, whence $x = 4$ or 12.

By substitution, we find that 4 is a root and 12 is not.

23. $\sqrt{x+21} = 7-\sqrt{x}$.

24. $\sqrt{x+5}+\sqrt{x} = 10/\sqrt{x}$.

Solve for x and y:

25. $2x+3y = 14$,
$x+y = 5$.

26. $7x-3y = 5$,
$5x-2y = 4$.

27. $\frac{3}{4}x-2y = 2$,
$x+8y = 24$.

28. $2.4x-1.6y = 1.04$,
$0.8x-2.5y = 0.74$.

29. $2x/7+5y/6 = 13/42$,
$x/7-10y = -27/14$.

30. $0.5x-3.1y = -10.3$,
$2.5x+4.5y = 8.5$.

Solve for x, y, and z:

31. $x+y = 5$,
$y+z = 7$,
$z+x = 6$.

32. $x+2y = 5$,
$2x-3y+z = 11$,
$x+5y-2z = 3$.

33. $x+y+z = 0$,
$2x-3y+z = -7$,
$3x+2y+z = -5$.

34. $2x+3y+4z = 3$,
$6x+6y+8z = 7$,
$x-y-z = -1/12$.

What conclusion regarding x can be drawn from each of the following inequalities?

35. $x-4>0$.

36. $x+4>0$.

37. $x+2<0$.

38. $-x<3$.

39. $4x>7$.

40. $-3x>6$.

41. $4x+5<3x+7$.

42. $8x>5x-9$.

43. $x^2>4$. **44.** $x^2\geqq 25$.

45. $-x>-4$. **46.** $-x\leqq 10$.

47. $x^2-49\leqq 0$. **48.** $9-x^2\geqq 0$.

SET No. 8: INDICES

SOME FACTS

$$a^m \cdot a^n = a^{m+n}; \qquad a^m/a^n = a^{m-n};$$
$$(a^m)^n = a^{mn}; \qquad a^{-m} = 1/a^m;$$
$$a^{1/m} = \sqrt[m]{a}; \qquad a^0 = 1;$$
$$(ab)^m = a^m b^m; \qquad (a/b)^m = a^m/b^m;$$
$$(\sqrt[n]{a})^m = a^{m/n} = \sqrt[n]{a^m}.$$

Give the meanings of the following:

1. x^{-5}. **2.** $a^{1/2}$. **3.** $b^{3/4}$. **4.** $m^{-1/4}$.

5. c^0. **6.** $n^{-6/5}$. **7.** r^{-100}. **8.** e^{-at}.

9. $x^{-2/3}$. **10.** k^{-1}. **11.** $p^{1/3}$. **12.** $y^{5/2}$.

Express by means of indices the following:

13. $\sqrt{5}$. **14.** $\sqrt[3]{10}$. **15.** $\sqrt[5]{2.4}$. **16.** $1/\sqrt{6}$.

17. $1/(\sqrt[3]{7})^2$. **18.** $1/\sqrt{5^3}$.

Solution:

$1/(\sqrt[3]{7})^2 = 1/7^{2/3} = 7^{-2/3}$

19. $\sqrt[5]{6^3}$. **20.** $\sqrt{a^3}$.

Evaluate:

21. $8^{1/3}$. **22.** $32^{1/5}$.

23. $8^{-1/3}$. **24.** 2^{-1}.

Solution: $8^{-1/3}=1/\sqrt[3]{8}=1/2$.

25. $125^{2/3}$.

26. $27^{2/3}$.

27. $256^{5/8}$.

28. $9^{-3/2}$.

29. $100^{-1/2}+81^{1/4}-(\sqrt{25})^0+(\sqrt{16})^{-1}+1/2^{-3}$.

30. $3^2+10^0-4^{1/2}-(1/25)^{-1/2}+0^4+(\sqrt{1/16})^{-2}$.

31. Divide $4x^{5/2}+3x^{3/2}-x^{-1/2}$ by $x^{1/2}$.

32. Multiply $x^{2/3}-x^{1/3}y^{1/3}+y^{2/3}$ by $x^{1/3}+y^{1/3}$.

33. Multiply $2x^{1/3}-1-3x^{-2/3}$ by $3x^{1/3}+2-2x^{-1/3}$.

34. Divide $9a^{4/5}-9a^{2/5}-25+23a^{-2/5}+6a^{-4/5}$ by $3a^{3/5}-4a^{1/5}-a^{-1/5}$.

35. Express by means of indices: (*a*) $\sqrt[3]{1+3x}$, (*b*) $\sqrt{(1+x)^3}$.

36. Use indices to write the following without denominators: (*a*) $1/\sqrt{1-x}$, (*b*) $1/\sqrt[3]{(1+2x)^2}$, (*c*) $1/(1+x)^3$.

37. Find the square root of $4x^2+x^{-2}+8x+4x^{-1}+8$.

38. Find the square root of

$$25x^{4/3}-30x^{2/3}+49-24x^{-2/3}+16x^{-4/3}.$$

39. Simplify:

$$(x^{1/2}+y^{1/2})^2-(x^{1/2}+y^{1/2})(x^{1/2}-y^{1/2})-(x^{1/2}-y^{1/2})^2.$$

40. Simplify: $(a^{1/3}+2)^3-(3a^{2/3}+4)(a^{1/3}+2)$.

41. Divide $x^{1/2}-2x^{1/4}+1-x^{-1/2}$ by $x^{1/4}-1-x^{-1/4}$.

42. Divide $x+2x^{5/6}-2x^{1/6}+2x^{-1/6}-2x^{-5/6}-x^{-1}$ by $x^{1/3}-x^{-1/3}$.

43. Evaluate: $64^{2/3}-16^0+(2^9)^{1/3}+2^{3/5}/2^{-2/5}-3(4/9)^{1/2}$.

44. Simplify: $\dfrac{2^n\cdot(2^{n-1})^n}{2^{n+1}\cdot 2^{n-1}\cdot 4^{-n}}$.

SET No. 9: GRAPHS

1. On a sheet of graph paper, draw axes of x and y, indicate a unit, and plot the following points: (5, 4), (—3, 2), (4, 0), (—4, —6), (0, 5), (3, —4), (—6, 6), (—2, 0), (3.4, —5), (0, —1), (—4.6, —2.8).

2. Name the quadrant in which each point of ex. 1 lies.

3. Under what condition or conditions will the point (x, y) lie (*a*) in the first quadrant? (*b*) in the second quadrant? (*c*) in the third? (*d*) in the fourth? (*e*) on the x-axis? (*f*) on the y-axis?

4. Plot the points $(-1, 1)$, $(-5, 1)$, and $(-1, 4)$. What is the point which, with these three points, makes up the vertices of a rectangle.

By finding three pairs of values for x and y, draw the graph of each of the following linear equations:

5. $x+y=4$.
6. $x-y=3$.
7. $2x=y+6$.
8. $x+2y=2$.
9. $3x-4y=12$.
10. $2x+3y=6$.
11. $2x-3y+5=0$.
12. $4x+3y+1=0$.
13. $4x-y-5=0$.

Construct tables of values and draw the graphs of the following equations:

14. $y=x^2$.
15. $x^2+y^2=25$.
16. $y=x^2-3x$.
17. $xy=2$.
18. $y+x^2=0$.
19. $x^2+y^2=4$.
20. $y=-x^2-4x$.
21. $y=(x-2)(x+1)$.
22. $x^2-y^2=4$.
23. $y=x^2-4x+3$.
24. $y=-(x+2)(x-1)$.
25. $y=x^2+1$.

Solve each of the following systems graphically:

26. $2x-y+5=0$, $3x+2y-3=0$.
27. $2x-3y=12$, $x+y=5$.
28. $3x-4y=0$, $x=4$.
29. $x^2+y^2=25$, $x+y=7$.
30. $y=x^2-4$, $y=2x-4$.
31. $xy=3$, $3x-y=0$.

ANSWERS TO EXERCISES

Page 4

2. (*a*) $x>2$; (*b*) $x\leqq 2$; (*c*) $-3<x<5$; (*d*) $-3\leqq x\leqq 5$; (*e*) $-6\leqq x\leqq 2$; (*f*) $a<x<b$; (*g*) $m\leqq x<n$.

3. (*a*) $-1\leqq x\leqq 1$ or $|x|\leqq 1$; (*b*) $-7<x<7$ or $|x|<7$;
   (*c*) $x\leqq -5$ and $x\geqq 5$ or $|x|\geqq 5$;
   (*d*) $x<-100$ and $x>100$ or $|x|>100$;
   (*e*) $x<-b$ and $x>b$ or $|x|>b$.

Page 6

1. $f(1)=0$; $f(0)=-2$; $f(-1)=-10$; $f(2)=8$; $f(-2)=-36$.

2. $G(a)=0$; $G(-a)=0$; $G(1/x)=x^{-2}-a^2$; $1/G(x)=1/(x^2-a^2)$.

3. $F(a)=0$; $F(-a)=-2a^3$; $F(2a)=7a^3$; $F(0)=-a^3$.

4. $f(2)=5$; $f(1/3)=-5$; $f(-y)=2/y-3y$; $f(1/y)=3/y-2y$; $1/f(y)=y/(3y^2-2)$.

5. $f(0)=1$; $f(-1)=3/2$; $f(3)=17$; $f(-z)=z^2+2^{-z}$; $f(1/z)=z^{-2}+2^{1/z}$; $1/f(z)=1/(z^2+2^z)$.

6. $P(x+1)=x^3+x^2+x+1$; $P(x-1)=x^3-5x^2+9x-5$; $P(2x)=8x^3-8x^2+4x$; $P(x^2)=x^6-2x^4+2x^2$.

7. $(1+x)/(1-x)$. 8. 0.

9. $f(3)=1080$; $f(5)=2430$; $f(5-r)=10(2^r\cdot 3^{5-r})$.

10. $\{F(x)\}^2=2^{2x}$; $F(x^2)=2^{x^2}$; $F\{F(x)\}=2^{2^x}$; $2^{2^{1/x}}$; $2^{2^{-x}}$.

11. $F(m-n)$; $F(mn)$; $F(m/n)$.

12. 0; 0; 9/2; 10/3; a^2-a-2.

13. $k=-7$; $m=12$; $L(-3)=33$.

14. $a=1$; $b=-8$; $P(4)=32$.

Page 16

1. (*a*) $n=59+m$; (*b*) $n=60+m$.

2. (*a*) $\sqrt{25-x^2}$; (*b*) $x\sqrt{25-x^2}/2$.

3. (*a*) $100/y$; (*b*) $\sqrt{y^4+10^4}/y$.

4. (*a*) $h=V/\pi r^2$; (*b*) $r=\sqrt{V/\pi h}$.

5. (*a*) $h=1600/s^2$; (*b*) $d=\sqrt{2s^6+40^4}/s^2$;
(*c*) $d=\sqrt{(3200+h^3)/h}$.

6. $A=120x-2x^2$ *or* $A=60y-y^2/2$.

7. $m=0.914y$. 8. $100+5m/12$.

9. (*a*) $F=32+9C/5$; (*b*) $C=5(F-32)/9$.

10. $l=3d/5$. 11. $f(x)=2x+1$.

12. $f(x)=x^2+4$.

Exercises on Chapter I—Page 18

1. $f(3)=0$; $f(-3)=0$; $f(0)=9$; $f(4)=-7$.

2. $f(0)=0$; $f(2)=4$; $f(4)=0$; $f(2-x)=4-x^2$.

3. $f(0)=2$; $f(1)=1$; $f(-2)=1/4$; $f(-4/3)=0$;
$f(1/x)=(3+4x)/(5+2x)$.

5. $A=5h$. 6. (*a*) $6h-h^2$; (*b*) $\sqrt{2h^2-12h+36}$.

10. $V=4x(6-x)^2$. 14. $(1+a^{-8x})/2$.

15. $1+x^2$.

16. $F(y)=y^2-y+1$; $F(x)=x^2-x+1$; $F(x-1)=x^2-3x+3$.

28. (*a*) $125/v$; (*b*) $4v/5$. 29. (*a*) 196 ft; (*b*) 2.5 secs.

37. $2x+h-4$; $2x-4$.

38. (*a*) $9x+16$; (*b*) $16+9/x$; (*c*) $4+3/(3x+4)$.

Page 30

1. The functions of (*a*), (*b*), (*c*), (*h*), (*i*), and (*j*) are linear; the others are not.
2. ax.
3. (*a*) 0.4096; (*b*) 7.5.
4. $6r-4$.
5. $2.4t+3.6$.
6. $(7x+5)/2$.
7. $1.6y-2.4$.
8. $35-9z$.
9. -4.
10. $2z+1$.
13. $7x-4$.

Page 37

3. $u=7v/2$.
4. 91/8.
5. 12.
7. $y=3x+5$.
8. 3/2 amps.; 2/3 amp.; 1/2 amp.
9. $5\frac{1}{3}$ in.; $4\frac{1}{2}$ pounds.
10. $v=8\sqrt{s}$; 160 ft. per sec.
11. 960 oz.
12. 5400 cu. ft.
13. 8/3 amps.
14. 5000/3 cu. cm.

Page 41

1. 2.50994.
2. 21.014.
3. 1.16472.
4. 0.59450.
5. 61.5°F.; 65°F.
6. 95795.

Exercises on Chapter II—Page 42

1. $\frac{3}{8}x-\frac{1}{2}$.
2. $7.5-4t$.
3. 29/3.
4. 15.
5. 8.
6. $2x$.
8. $50+\frac{3}{4}n$.
10. 17.5 lb. per sq. in.; 210 cu. in.
11. 151.6 lb. approx.
16. $39-11x/2$.
18. $-1.5x+7.5$.
19. $\frac{1}{2}ax+a$.
22. $y=1+2x+6/x$.
24. 8 ft. per sec.
25. 2:3.
26. 6 in.
28. 1350 h.p.
29. 0.26 in.
30. $30-15\sqrt{2}$ ft. (8.8 ft. app.)

33. 5.2550. 34. 17.4 in. 35. 0.085:1.

36. 27 lb. very nearly. 37. 4000 miles.

38. 67 million miles. 39. $6\frac{1}{4}$ kgm.

Page 54

1. The functions of (*a*), (*b*), (*c*), and (*i*) are quadratic; the others are not.

3. Maximum 6.25. 8. Axis of symmetry: $x=2/3$.

9. Axis of symmetry: $x=7/8$. 10. $p<0$, $q>0$.

11. $b=0$. 13. $2r^2+r-15$.

14. $-3x^2+5x+4$. 15. x^2-x.

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2. (*a*) $4/5, -3/2$; (*b*) $2/7, -6$; (*c*) $3/7, -5/2$; (*d*) $0.3, -1.2$.

3. (*a*) $-1\pm\sqrt{3}$; (*b*) $(-6\pm\sqrt{6})/5$; (*c*) $(5\pm3\sqrt{5})/10$; (*d*) $6(1\pm\sqrt{3})/5$.

4. (*a*) $\sqrt{6}/2$; (*b*) $(7\pm\sqrt{17})/4$; (*c*) $(-q\pm\sqrt{q^2-pr})/p$; (*d*) $a(1\pm\sqrt{5})/2$.

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2. $1, \sqrt{-1}, -1, -\sqrt{-1}$.

3. $-\frac{2}{3}i$, $7-24i$, $2+11i$, $\frac{1}{2}-\frac{1}{2}i$, $\frac{3}{5}+\frac{4}{5}i$.

4. (*a*) 13; (*b*) $-1+3i\sqrt{6}$; (*c*) -10.

5. (*a*) $-3-6i$; (*b*) $1+7i$; (*c*) $2+3i$.

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1. (*a*) real and unequal; (*b*) imaginary; (*c*) real and equal.
2. (*a*) $x^2+6x-16=0$; (*b*) $6x^2-17x-14=0$;
   (*c*) $x^2-(2m+3n)x+6mn=0$; (*d*) $x^2-2ax+a^2-b^2=0$.
3. (*a*) $x^2+4x+8=0$; (*b*) $x^2-10x+21=0$;
   (*c*) $2x^2-11x+30=0$; (*d*) $x^2-3bx+2b^2=0$.
5. (*a*) (i) $-2\pm\sqrt{3}$, (ii) $\frac{1}{2}\pm i$;
   (*b*) $(x+2+\sqrt{3})(x+2-\sqrt{3})$, $(2h-1+2i)(2h-1-2i)$.
6. (*a*) $3(x+i\sqrt{7/3})(x-i\sqrt{7/3})$; (*b*) $(x-2-\sqrt{5})(x-2+\sqrt{5})$;
   (*c*) $-(x-\frac{1}{2}+\frac{1}{2}i\sqrt{3})\,(x-\frac{1}{2}-\frac{1}{2}i\sqrt{3})$; (*d*) $(x-3)(4x-5)$.
7. (i) $6x^2-8x+5=0$; (ii) $36x^2-4x+25=0$.
8. (*a*) p^2-q; (*b*) $p/(6p^2-25q)$; (*c*) $(p^2-2q)/q$.
9. (*a*) $px^2+(q-10p)x+r-5q+25p=0$;
   (*b*) $p^3x^2+(q^3-3pqr)x+r^3=0$;
   (*c*) $p^2x^2+4rp-q^2=0$.

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1. (*a*) $-4<x<5/3$; (*b*) $x<0$ and $x>10/3$; (*c*) none.
2. (*a*) $x<-2/3$ and $x>3$; (*b*) none; (*c*) $-5/2<x<5/2$.
3. (*a*) positive for all real values of t; (*b*) vanishes when $x=\sqrt{6}/3$, otherwise positive; (*c*) negative, except when $-4/3\leqq p\leqq 5/2$.
4. (*a*) 4, 5, 6; (*b*) all integral values of z except 0;
   (*c*) none.
5. (*a*) $-7<y<3$; (*b*) no real values; (*c*) all values except those for which $(-1-\sqrt{17})/4\leqq t\leqq(-1+\sqrt{17})/4$.
6. $\frac{8}{3}x^2-12x+\frac{40}{3}$.

7. (*a*) $r=0$ gives min. -9; (*b*) $r=0$ gives max. 9;
(*c*) $x=7$ gives min. 4; (*d*) $x=7$ gives max. 4;
(*e*) $z=-3$ gives min. -32; (*f*) $z=-3$ gives max. 32.

8. (*a*) 4, when $x=0$; (*b*) $-\frac{1}{4}$, when $x=-\frac{1}{2}$;
(*c*) b^2, when $x=b$.

9. (*a*) 8, when $t=0$; (*b*) 7, when $t=2$;
(*c*) 25/12, when $t=5/6$.

10. 2500 sq. rods. **11.** 3 in. **12.** 2.5 feet. **13.** 1100.

Exercises on Chapter III—Page 84

1. (*a*) $2\pm 2i$; (*b*) 21/2, 5/3; (*c*) $-9/5$, $-9/5$.

2. (*a*) $x^2+x-6=0$; (*b*) $20x^2-x-1=0$; (*c*) $x^2-r^2=0$;
(*d*) $x^2-2mx+m^2-1=0$.

3. (*a*) imaginary; (*b*) real and equal;
(*c*) real and unequal; (*d*) imaginary.

4. (*a*) $4(x+\frac{1}{2})^2+8$; (*b*) $3(x-\frac{2}{3})^2-\frac{4}{3}$;
(*c*) $(x+m/2)^2+(4n-m^2)/4$.

5. $3x^2-5x+7$. **6.** (*a*) $m\leqq 25/4$; (*b*) $|m|\geqq 12$; (*c*) $m\leqq 1$.

9. $2x^2-5x+8$.

10. (*a*) $7+24i$; (*b*) $-44+117i$; (*c*) $4/25-3i/25$.

11. (*a*) min. 2, when $x=3$; (*b*) max. 81/8, when $x=-5/4$;
(*c*) min. 31/8, when $x=5/4$.

13. (*a*) 80/3; (*b*) 343. **14.** 50, 50.

15. 1/2. **16.** $5x^2-18x+7$.

17. (*a*) $(x+i)(x-i)$; (*b*) $\frac{1}{4}(2x+1+i\sqrt{3})(2x+1-i\sqrt{3})$;
(*c*) $(2x+5-\sqrt{6})(2x+5+\sqrt{6})$.

19. $x^2+2x+3=0$.

20. (*a*) $p \leqq 0$; (*b*) $p \leqq 0$ and $p \geqq 4$; (*c*) $-1 \leqq p \leqq 1$.

21. $-3(x^2-6x+5)$. **22.** $p^2-4q=0$.

24. (*a*) $9x^2-73x+16=0$; (*b*) $27x^2+595x-64=0$;
(*c*) $27x^2-84x-64=0$.

25. (*a*) $a>0$; (*b*) $c=0$; (*c*) $b^2-4ac=0$;
(*d*) a large numerically.

26. (*a*) $c=a$; (*b*) $2b^2=9ac$. **29.** $12.

30. 60 rods by 40 rods. **31.** 60 rods by 30 rods.

32. 156.25 ft; 25/8 secs. **35.** $a:b:c=p:q:r$.

36. $(aq-bp)(br-cq)=(ar-cp)^2$.

38. All real values except those between 5 and 9.

43. $f(x) \leqq 49/20$. **46.** All real values. **47.** 3.

51. m cannot have values between $-5/12$ and $12/5$.

52. $-8/7$.

53. (*b*) $Q(x)=\frac{1}{2}(x-2)(x-3)-\sqrt[5]{2}(x-3)(x-1)$
$+\frac{1}{2}\sqrt[5]{3}\,(x-1)\,(x-2)$.

55. 15′x10′.

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1. The functions of (*a*), (*c*), (*e*), and (*f*) are polynomials; the others are not.

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2. (*a*) -7; (*b*) 5; (*c*) -100.

3. (*a*) -125; (*b*) 0; (*c*) -100.

6. (*a*) 44; (*b*) 87; (*c*) $10\frac{19}{27}$.

11. (*a*) 3, -15; (*b*) 11/2. **12.** 1.

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2. (*a*) $x(x-3)(x+3)$; (*b*) $(x-1)(x+1)(x-2)(x+2)$;
(*c*) $(x-3)(x+3)(x-i)(x+i)$.

3. $(x-2)(x+2+i)(x+2-i)$.

4. $p=a,\ q=b-2a,\ r=a-b+c$.

6. $2(x+1)(2x-1)(x-2)$.

7. (*a*) -12; (*b*) 2.

9. $(x^2-1)(x-3)$.

10. $4(x-1)^3+6(x-1)^2-3(x-1)+5$.

11. $(z+2)^3-6(z+2)^2+12(z+2)$.

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1. $3, 2, -2, -1$.

2. $-1, -6, (-7\pm\sqrt{-15})/2$.

3. $2, -2, \pm\sqrt{7/3}$.

4. 1 (double root), -4.

5. $-1, -5/3, 4$.

6. $3, -1\pm i$.

7. $2/3, 3/2, 2, 1/2$.

8. $0, 2, 1/2, -2$.

9. $(-1\pm\sqrt{34})/3, (-1\pm\sqrt{2})/3$.

10. $2\pm\sqrt{3}, (11\pm\sqrt{-23})/12$.

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2. $x^2-6x+25=0$.

3. $(3x-2)(x^2-4x+5)=0$.

4. $(x^2+16)(x^2-4x+20)=0$.

5. $1\pm i, 2, -2$.

6. $3\pm i\sqrt{2}, 3, 1/3$.

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1. (*a*) Quotient $x^2-8x+21$, remainder -46;
(*b*) quotient $2x^3+6x^2+21x+62$, remainder 184;
(*c*) quotient x^3+2x^2+4x+8, remainder 20;
(*d*) quotient $3x^2-18x+103$, remainder -616.

2. (*a*) 388; (*b*) 172; (*c*) -4.

3. -210; 15.

4. $-155/16$; 5690.

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1. (*a*) Between -4 and -3, -1 and 0, 1 and 2; (*b*) between 1 and 2; (*c*) between -2 and -1; (*d*) between -1 and 0.

2. 1.39. 3. 2.07. 4. 1.44. 5. 3.27.

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1. 0.73. 2. 0.77 *r*.

Exercises on Chapter IV—Page 121

1. (*a*) 32; (*b*) 1; (*c*) 1. 2. $a=3$, $b=-13$.

3. (*a*) $x^3+x^2-6x=0$; (*b*) $(x+4)(9x^2+12x+5)=0$; (*c*) $(x-2)^4-9=0$.

4. $1\pm\sqrt{2}$.

5. (*a*) $3, 1, \frac{1}{2}, -\frac{3}{2}$; (*b*) $(x-3)(x-1)(2x-1)(2x+3)$.

7. $x^3+3x^2-4=0$. 8. $-3\pm\sqrt{5}$.

9. $(x-1)(x^2-6x+13)=0$. 10. $\pm\sqrt{15}$, $\pm 2i$.

11. $0, (1\pm i\sqrt{3})/2$. 13. $a=3/2$; $p=-5/2$; $c=-2$.

14. All values. 15. $2(x+1)(x-1)(x-4)$.

17. (*a*) $-a$; (*b*) $-c$.

18. $a=-(k+l+m+n)$; $b=kl+km+kn+lm+ln+mn$; $c=-(klm+kln+kmn+lmn)$; $d=klmn$.

19. $(t+1)^3-2(t+1)^2+2(t+1)$. 20. $2(x^2-1)(x^2-4)$.

21. $4(x-1)(x+2)(x-2)$. 22. $1, (5\pm\sqrt{13})/2$.

23. $1, 1, -3\pm 2\sqrt{2}$. 24. $-1\pm i\sqrt{2}, (-5\pm\sqrt{37})/6$.

25. $(-3\pm\sqrt{241})/4, (-3\pm\sqrt{-135})/4$.

26. $1, 2, (-5\pm\sqrt{17})/2$.

27. $1, 1, 1, 3$.

28. -2.

31. $1, -3, (-3\pm\sqrt{-7})/4$.

36. $f(2)=0$; $f(-1)=0$; $f(-2)=-20$; $f(-3)=0$.

38. Between -4 and -3, 0 and 1, 4 and 5; 4.26.

39. -1.80; -0.45; 1.24.

40. $7.56''$.

41. $1.27''$.

42. $3.78''$.

47. $x(x^2-1)(x+3)$.

48. $x^3-3x^2-6x-20=0$.

49. $2p^3-9pq+27r=0$.

50. $a=2$, $b=3$.

51. $(-5\pm\sqrt{-11})/2$, $(-5\pm\sqrt{17})/2$.

53. $a=5$; $b=2$ or -2; $c=71$.

54. $a=6$; $b=7$.

56. $2\pm\sqrt{3}$, 1, -1.

57. -3, $4/3$, $5\pm\sqrt{2}$.

58. (*a*) quotient x^2-4x-6, remainder $-3x+16$;
(*b*) quotient $2x^2-x+2$, remainder 0;
(*c*) quotient x^3-x^2+1, remainder $-x+3$.

59. 0.93 ft.

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1. The functions of (*a*), (*d*), and (*g*) are polynomials; the others are not.

6. (*a*) $\{x+y(\sqrt{2}+1)+\sqrt{2}\}\{x+y(1-\sqrt{2})-\sqrt{2}\}$;
(*b*) $(x+y-2)(x-y+4)$; (*c*) $(x+4y-2)(x-2y+4)$.

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1. The functions of (*a*), (*b*), (*d*), and (*g*) are rational; the others are not.

2. The functions of (*a*), (*c*), (*e*), and (*f*) are rational; the others are not.

3. (*a*) $\dfrac{4x^2+14x+15}{x^2+4x+4}$; (*b*) $\dfrac{2x^2+8}{2x}$; (*c*) $\dfrac{3x^2-6}{x^3+2x}$.

5. About 0.6.

7. (*a*) $\dfrac{2}{x-1}+\dfrac{1}{x-2}$; (*b*) $\dfrac{1}{x+1}+\dfrac{1}{x-1}$; (*c*) $\dfrac{3}{x}-\dfrac{2}{x+1}$.

8. (*a*) $\dfrac{3}{x-3}-\dfrac{2}{x+3}$; (*b*) $\dfrac{5}{2x-1}-\dfrac{2}{x+2}$.

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1. (*a*) 16 : 3; (*b*) 7 : 40; (*c*) 7 : 3.

2. \$$128\frac{1}{3}$ and \$$201\frac{2}{3}$.

3. (*a*) 2 : 3; (*b*) $a/(a+b)$, $b/(a+b)$.

4. (*a*) $58\frac{2}{3}$ ft. per sec.; (*b*) 35,280 metres per min. per min.

5. (*a*) 8; (*b*) 12; (*c*) $r+s$. **11.** 50, 75, 100.

12. 5. **13.** $x : y = -5 : 2$.

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2. (*a*) 2 : 3 : 4; (*b*) 1 : 3 : 5; (*c*) $-m : 2m^2+3 : 2+m^2$.

3. (3, 8), (8, 3).

4. $(\pm\sqrt{10}, \pm\sqrt{10})$, $(\pm 6\sqrt{5/13}, \pm 4\sqrt{5/13})$.

5. (1, −1), (−1, 1), (6, 1), (−12/37, −2/37).

6. (4, 3), (−80/17, −67/27). **7.** (4, −1), (−1, 4).

8. (8, −4), (10, 0). **9.** (2, 1), (−2, −1).

10. (3, 2, 1), (−3, −2, −1). **11.** (2, −3, 4), (−2, 3, −4).

12. 120 rods by 50 rods. **13.** 1, 2, 4.

Exercises on Chapter V—Page 152

1. (*a*) $4:3$; (*b*) $2:9$. **2.** (*a*) 6; (*b*) a^2b.

3. (*a*) $-15/2$; (*b*) q^4.

4. $(5, 8), (-5, -8), (2, -1), (-2, 1)$.

5. $2:7:3$. **8.** 9. **9.** 14′ and 21′.

11. $(2, -2)$. **12.** $5\frac{2}{7}, 5\frac{2}{7}, 3\frac{3}{7}$. **14.** $(2, 1)$.

17. $(7, -4), (4, -7), (-8, 1), (-1, 8)$.

18. $(3, 2), (2, 3), (-4\pm\sqrt{10}, -4\mp\sqrt{10})$.

21. $(2, 6, -4), (-2, -6, 4)$.

22. (*a*) $\dfrac{3}{x+3}+\dfrac{2}{x-2}$; (*b*) $\dfrac{5}{2x+1}-\dfrac{2}{x-2}$.

23. 50 cm. by 20 cm. **24.** -2 or -10.

25. $(3, 2), (-3, -2), (2, 3), (-2, -3)$.

26. Increase length by 16.18 rods and decrease width by 1.18 rods; decrease length by 6.18 rods and increase width by 21.18 rods.

36. $(2, 3), (6, 1), (3, 2), (4, 3/2)$.

37. (*a*) $r<2$ and $r>3$; (*b*) $r=3$ or 2; (*c*) $2<r<3$.

38. $a+b+c=0$; $x=\dfrac{ab-c^2}{ac-b^2}$, $y=\dfrac{cb-a^2}{ac-b^2}$.

40. 8, 12, 18, 27; 27, 18, 12, 8; 27, 12, 18, 8; 8, 18, 12, 27.

41. $(5, -2, 1), (-2, 5, 1)$. **42.** $(3, 4, 5), (-13, -14, -15)$.

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2. (*a*) $7+10+13+16$; (*b*) $2+2\frac{1}{2}+3\frac{1}{3}+4\frac{1}{4}$;
(*c*) $\log 2+\log 3/2+\log 4/3+\log 5/4$; (*d*) $6+12+24+48$;
(*e*) $1+1/4+1/9+1/16$.

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3. 161; $4k+1$.
4. -15, -195, -1995, $5-2m$.
5. 1, 2/3, 21, $(2x+1)/3$.
6. -17.2; $-1.2x+2$.
7. 30.
8. t_{20}.
9. 17.
10. 1; -6.
11. -3; 3.
12. $a=7$, $d=-3$; $10-3p$.
13. 7.
14. $k-2n+2$; $k-76$, $k-98$, $k-198$.
15. 4, 9, 14, 19, $\cdots$, 84, 89, 94, 99.
16. 10.
17. t_{100}.
18. $205b$, $b(4n+3)$.

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1. (a) 9; (b) $2\sqrt{5}$; (c) x.
2. 11, 15, 19, 23, 27.
3. $2/\sqrt{2}$, $3/\sqrt{2}$, $4/\sqrt{2}$, $5/\sqrt{2}$.
4. $2k-a$, $3k-2a, \cdots, 8k-7a$.
5. 32′, 44′, 56′, 68′.

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1. (a) 650; (b) -2150.
2. 3775.
3. (a) $p^2/10$; (b) -555.
4. -1650; $-11{,}625$; $5(7x-x^2)/4$.
5. 4.5, 6, 7.5, 9, 10.5, 12, 13.5; 63.
6. 50,050.
7. 60; 6600.
8. 476.
9. 2850.
10. $d=-5$; $t_6=-1$.
11. $n\{r(2n+11)+s(2n-1)\}$.

12. 101.

13. 7 or 10.

14. 69.

15. 156.

16. $dr^2+r(6c-d)$.

17. $n(25-5n)/2$; -375.

18. 32; 4.

19. 8.

20. $135\frac{5}{12}$ ft.

21. 25 sec.

22. 129.

23. 10.

24. 1140 feet.

25. 320.

26. (*a*) 5; (*b*) 9.

27. -2400.

28. $\dfrac{p(m-r+1)+rq}{m+1}$.

30. 33 secs.; 1496 feet.

31. 220.

34. 5; $n=1, 2, \cdots, 7$.

35. $\dfrac{l-a}{n-1}$; $\dfrac{2(s_n-an)}{n(n-1)}$.

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1. (*a*) 3/2, 6/5; (*b*) 50, $42\frac{6}{7}$; (*c*) $-3/4$, $-3/5$.

2. (*a*) 20/3; (*b*) $(a^2-b^2)/a$.

3. (*a*) -60, -40, -30; (*b*) $\frac{5}{6}$, $\frac{10}{13}$, $\frac{5}{7}$, $\frac{2}{3}$, $\frac{5}{8}$, $\frac{10}{17}$, $\frac{5}{9}$, $\frac{10}{19}$.

4. $\frac{1}{4}$; $\frac{87}{40}$.

5. $-4/15$.

6. 120, 60, 40, 30, 24, 20.

7. 2, 1, 2/3, $\cdots$; *or*
2, 20/3, -5, $\cdots$.

8. 15 and 3.

10. 1 : 4.

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3. 1/9.

4. 5^{99}; 5^{k-1}.

5. $-1/8$; $(-2)^{7-p}$.

6. m^{41}; m^{201}; m^{2a+1}.

7. The 10th.

8. 2; $2\sqrt{2}$

9. 3.

10. 972, ±324, 108, $\cdots$.

11. 1; 4.

12. $16x^5$; $512x^{10}$; t_{11}.

13. (*a*) 5/128; (*b*) $3(10^{-11})$.

14. t_7.

15. 18,000; 500.

16. $a=15$; $r=5$.

17. $(m+5)(2m)^{p-1}$.

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1. (*a*) 9; (*b*) −1/4; (*c*) 1; (*d*) no real geometric mean.

2. (*a*) $\pm\frac{27}{32}, \frac{9}{16}, \pm\frac{3}{8}, \frac{1}{4}, \pm\frac{1}{6}$;
(*b*) ±6, 12, ±24, 48, ±96, 192, ±384.

3. 10, −4/5.

4. $h\left\{\frac{k}{h}\right\}^{\frac{p}{m+1}}$.

5. 3.

7. 14.9%.

8. 0.

9. 25%.

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1. (*a*) 15, 624; (*b*) −1023.

2. (*a*) 265,720; (*b*) $1\frac{255}{256}$.

3. $(1-x^{20})/(1+x)$.

4. $(128p^7-1)/(2p-1)$.

5. 3415.

6. $q>p$.

7. $r=3$ or -3; .1, .3, .9, or −.2, .6, −1.8.

8. $\frac{256}{6561}$.

9. $\frac{1261}{2187}$.

10. $\frac{(1+x)(1-x^ky^k)}{1-xy}$.

11. $89\frac{7}{32}$ feet.

12. $10,485.75.

13. 409.51 cm.

14. $120+40\sqrt{3}$.

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2. (*a*) 45/99; (*b*) 107/45; (*c*) 299/2475.
3. $r=-3/4$; 16, −12, 9, · · · .
4. $|x|<\frac{1}{3}$.
5. 1000 cm.
6. 112 feet.

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1. 7/9.
3. $100(1.03)^{28}$; $100(1.03)^{n+8}$.
4. −5460; $15\{1-(-3)^n\}/2$.
5. ±9604.
6. 25, 3; −50, −3.
7. 160, 80, 40, 20, 10, 5; 315.
8. (*a*) 4/9; (*b*) 5/33.
9. 6.
10. 0.015625.
11. 5.
12. $\sqrt{mn}$.
13. ±2.
14. $(p^{11}-q^{11})/(p-q)$.
15. 2, 10, 50, 250, · · · .
16. $2^{64}-1$; 20 digits.
17. Sum $=128\,(1-1/2^{13})$.
18. $a=32$, $r=1/4$; or $a=2$, $r=4$.
19. 19,680.
20. 5.
21. 162.
23. $a=9$, $r=2/3$.
24. $1-1/k$.
25. 7.
26. $a=15$, $r=5$.
27. 126.

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1. (*a*) $\frac{n}{3}(8n^2+21n+16)$; (*b*) $\frac{n}{2}(2n^2+5n+1)$;

(*c*) $\frac{n}{2}(6n^2+15n+11)$.

2. 5950.

3. $\dfrac{n}{6}(n+1)(4n-13)$; $s_7=140$.

4. 4.

5. 20,785.

Exercises on Chapter VI—Page 199

1. (*a*) -270; (*b*) $2\frac{249}{420}$; (*c*) 16; (*d*) 605; (*e*) 166.666665.

2. (*a*) -40, 960, $20(-2)^{p-1}$; (*b*) -12, $\dfrac{60}{7-p}$;
(*c*) $u-30v$, $u+v(6-3p)$; (*d*) 1645, $12p^2-7p+1$.

3. 65,534.

4. 189; 21.

5. 19/990.

6. -13, -12, $-\frac{144}{13}$.

7. 23 hours; 21 hours.

8. $2n(n+1)^2$.

9. 3.

10. a^{30}; a^{2n-12}.

11. 1280.

12. 3.

13. 5, 8, 11, 14.

14. 35,712; 508,860.

15. 4780 yd.

18. 1/4, 7/8, $-7/12$ or 1/4, 1/6, 1/8.

19. 5, 15, 45 or 65/7, $-195/7$, 585/7.

21. $t(7t+11)/2$.

22. $9^6/10^4$ gals.

25. $(B/A-1)^{1/n}$.

27. 1800 feet.

28. $\dfrac{3}{2}(3^r-1)-r(r+1)$.

29. $\dfrac{a-rl}{1-r}$.

31. 2, 4, 8, 16.

32. (*a*) 38; (*b*) 2,400,020,000.

37. $(2\pi-4)/\pi$; $\pi k(4+2\sqrt{2})$; $8k(1+\sqrt{2})$.

38. $\frac{1}{6}n(n+1)(n+2)(9n+23)$.

39. $\frac{1}{6}n\{2d^2n^2+3n(2ad-d^2)+6a^2-6ad+d^2\}$.

42. $\dfrac{1+x-(2n+1)x^n+(2n-1)x^{n+1}}{(1-x)^2}$; $\dfrac{1+x}{(1-x)^2}$.

44. $kp+q\,(2^k-k-1)$.

45. $(k+1)d$ or $(1-k)d$.

48. $2a^3-9ab+27c=0$.

49. 1, 3, 9.

50. $\dfrac{ar^{n+1}+r^n(d-a)-r(a+nd)+a-d+nd}{ar^{n-1}(r-1)^2}$.

Page 210

1. 40,320.
2. 720; 480; 240.
3. 676; 13,800.
4. 120; 216.
5. 2160.
6. 3024.
7. 1,860,480.
8. 2880.
9. 14,400.
10. 2730; 182.
11. 362,879.
12. 90,000; 1000; 52,488; 18,000.
13. 20.

Page 216

1. 504; 95,040; $m(m-1)$; 5040.
2. 8.
3. (*a*) 362,880; (*b*) 362, 880; (*c*) 3024.
4. 151,200.
5. (*a*) 362,880; (*b*) 80,640.
6. 12.
7. 60.
9. 288.
10. (*a*) 1440; (*b*) 3600.
11. 15,600; 7,893,600.
12. (*a*) 100,000;
    (*b*) 30,240; 69,760.
13. 720.

Page 221

1. 84; 84; $\frac{1}{6}k(k-1)(k-2)$; 1.
2. (*a*) 10; (*b*) 10; (*c*) 20.
3. 35.
4. 210.

5. 45; 120; 210; 210.
6. 126; 56; 70.
7. 7,054,320.
8. 4845; 4950.
10. 5,598,527,220.
11. 21.
12. 4800.
13. (*a*) 100; (*b*) 80.

Page 223

1. (*a*) 720; (*b*) 120.
2. 181,440.
3. 120.
4. 2450.
5. 30,030.
6. 190.
7. 6720.
8. (*a*) 20; (*b*) 15.
9. 360; 120; 240; 12.
10. (*a*) 56; (*b*) 252.
11. 3168.
12. 864.
13. 3360.
15. 90; 18; 72.
16. 170; 10.
17. 1,481,760.
18. 96.
19. 28,800.
20. 809,100; 9900.
21. 2,903,040.
22. 4,838,400.
23. 40,319.
24. 176,904,000.
25. 113,400.
26. 1140; 171.

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1. 45,360.
2. 25,200.
3. 554,400; 15,120.
4. 210.
5. 126.
6. 277,200; 15,120; 120,960.
7. 35.
8. 60.

Page 230

1. 399.
2. 255.
3. 15.
4. 10; 58.
5. 502.
6. 29.

Exercises on Chapter VII—Page 236

1. (*a*) 6; (*b*) 1.
3. 182.
4. (*a*) 60,480; (*b*) 1044.
5. 1716.
6. 14.
7. 210.
8. 756.
9. 1023.
10. 720.
11. (*a*) 53; (*b*) 758.
12. 6,110,599.
13. 165.
14. 85.
15. (*a*) 45; (*b*) 630.
16. 60.
17. 13,685,760.
18. (*a*) 18,145; (*b*) 14,535.
19. 126.
20. 25,410.
21. 935,298.
22. 369,600; 15,400.
23. 73,260.
24. $\frac{4}{3}k(k-1)(k-2)$.
25. 56.
26. 4095.
27. (*a*) 311; (*b*) 17,585.
29. 120.
30. (*a*) 120; (*b*) 4320.
31. 130.
32. 31; 16; \$608.
33. 5; 252.
34. $\frac{(m+n-2)!}{(m-1)!\,(n-1)!}$.
35. (*a*) 370; (*b*) 14,440.
36. $\frac{(m+1)!}{p!\,(m-p+1)!}$.
37. $\frac{13!\,47!}{8!}$.

38. (a) $\frac{(n-1)!}{(r-1)!\,(n-r)!}$; (b) $\frac{(n+r-1)!}{n!\,(r-1)!}$; $\frac{n!\,(n-1)!}{(r-1)!\,(n-r)!}$.

39. 7560. 41. (a) $\frac{(4b)!}{b!b!b!b!}$; (b) $\frac{(4b)!}{4!b!b!b!b!}$.

42. 7875. 43. 5796. 44. 300.

Page 251

5. $a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.
6. $m^4+4m^3n+6m^2n^2+4mn^3+n^4$.
7. $a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$.

Page 255

1. $1+6m+15m^2+20m^3+15m^4+6m^5+m^6$.
2. $1+10a+40a^2+80a^3+80a^4+32a^5$.
3. $1+7p+21p^2+35p^3+35p^4+21p^5+7p^6+p^7$.
4. $1-12x+54x^2-108x^3+81x^4$.
5. $1-5s^2+10s^4-10s^6+5s^8-s^{10}$.
6. $16+128x+384x^2+512x^3+256x^4$.
7. $m^6+6m^5n+15m^4n^2+20m^3n^3+15m^2n^4+6mn^5+n^6$.
8. $64+384a+960a^2+1280a^3+960a^4+384a^5+64a^6$.
9. $243-2430x+9720x^2-19{,}440x^3+19{,}440x^4-7776x^5$.
10. $1+3a+\frac{27}{8}a^2+\frac{27}{16}a^3+\frac{81}{256}a^4$.
11. $27-162x^2+324x^4-216x^6$.
12. $a^6+6a^6r+15a^6r^2+20a^6r^3+15a^6r^4+6a^6r^5+a^6r^6$.
13. $1-7a^2+21a^4-35a^6+35a^8-21a^{10}+7a^{12}-a^{14}$.
14. $x^{10}-5x^{11}+10x^{12}-10x^{13}+5x^{14}-x^{15}$.

15. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

16. $k^3-6k^4m^{-1}+15k^5m^{-2}-20k^6m^{-3}+15k^7m^{-4}-6k^8m^{-5}+k^9m^{-6}$.

17. $z^{15}-5b^2z^{12}+10b^4z^9-10b^6z^6+5b^8z^3-b^{10}$.

18. $1+4a^{5/4}+6a^{5/2}+4a^{15/4}+a^5$.

19. $x^6-6x^7+15x^8-20x^9+15x^{10}-6x^{11}+x^{12}$.

20. $1-40x^2+780x^4-9880x^6+\cdots$.

21. $m^{-17}+17m^{-15}+136m^{-13}+680m^{-11}+\cdots$.

22. $1+0.3+0.0405+0.00324+\cdots$.

23. $1+100\sqrt{2}m+9900m^2+\cdots$.

24. $a^{-10}-20a^{-9}+190a^{-8}+\cdots$.

25. $x^{24}-12x^{22}y^2+66x^{20}y^4+\cdots$.

26. $16+256\sqrt{2}+3584+\cdots$.

27. 161,051; (*a*) 96,059,601; (*b*) 104,060,401.

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2. ${}_{1000}C_{49}x^{49}$; ${}_{1000}C_{32}x^{32}$; ${}_{1000}C_{20}$; ${}_{1000}C_{75}$.

3. $-{}_{100}C_{49}x^{49}$; ${}_{100}C_{32}x^{32}$; ${}_{100}C_{20}$; $-{}_{100}C_{75}$.

4. ${}_{30}C_r y^r$.

5. ${}_{15}C_r(-3x)^r$.

6. $5^{20}\cdot{}_{20}C_r(-2x)^r$.

7. ${}_{13}C_r\, a^{13-r}x^{2r}$.

8. ${}_{10}C_r\, a^{30-5r}$.

9. $(-3)^r{}_{24}C_r n^{3r-24}$.

10. 210.

11. (*a*) t_4; (*b*) none.

12. $(-1)^r{}_{20}C_r x^{20-3r}$; $-1140x^{11}$; $-1140x^{-31}$; t_7.

13. $-55x^{18}/512$.

14. 70; $28n^4/m^4$.

15. ${}_{100}C_r\cdot m^r\cdot 2^{r/2}$.

16. $(-1)^r \cdot {}_{10}C_r \cdot 2^r \cdot z^{3r+20}$; $-8064z^{35}$; $-960z^{29}$; -15360.

17. (*a*) $3360c^{12}$; (*b*) $-1140s^{11}k^{-31}$; (*c*) $3584x^{18}$.

18. 15; ${}_{21}C_{14}$.

19. (*a*) ${}_{16}C_8 \cdot k^{-16}$; (*b*) $-{}_{16}C_{13} \cdot k^{-36}$; k^{-8} in t_7; k^5 does not occur.

20. $(-1)^r \cdot {}_6C_r \cdot n^{3r-6}m^{6-2r}$; t_5; $-20n^3$.

21. ${}_{30}C_{29-r} \cdot a^{1+r}(-b)^{29-r}$; ${}_{30}C_{28} \cdot a^{30-28}b^{28}$;
${}_{30}C_{x+7} \cdot a^{23-x}(-b)^{x+7}$; ${}_{30}C_{n-2} \cdot a^{32-n}(-b)^{n-2}$.

22. $m+k=52$; 4 or 46.

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1. $1+\frac{3}{4}x-\frac{3}{32}x^2+\cdots$; $|x|<1$.
2. $1+x/2+3x^2/8+\cdots$.
3. $1-8a+40a^2+\cdots$; $|a|<1/2$.
4. $1-3p/2-9p^2/8+\cdots$.
5. $2+2x-x^2+\cdots$; $|x|<1/2$.
6. $27-81a+81a^2/2+\cdots$.
7. $1/2+x+2x^2+\cdots$; $|x|<1/2$.
8. $4-2x/3-x^2/36+\cdots$.
9. $1+2x^2+3x^4+\cdots$; $|x|<1$.
10. $1/4+3k/4+27k^2/16+\cdots$.
11. $1/3-a/27+2a^2/243+\cdots$; $|a|<3$.
12. $1-5x^2+15x^4+\cdots$.
13. $(-1)^{r-1}\dfrac{3\cdot 1\cdot 5\cdots(4r-7)}{4^r\cdot r!}\cdot x^r$.
14. $\dfrac{1\cdot 3\cdot 5\cdots(2r-1)}{2^r\cdot r!}\,x^r$.

15. $\dfrac{(r+1)(r+2)(r+3)}{6}(-2a)^r.$

16. $\dfrac{-1\cdot 1\cdot 3\cdots(2r-3)}{2^r\cdot r!}\cdot 3^r p^r.$

17. $\dfrac{-2\cdot 3\cdot 5\cdots(2r-3)}{r!}(-x)^r.$

18. $\dfrac{81\cdot 1\cdot 3\cdots(2r-5)}{r!}a^r.$

19. $2^{r-1}x^r.$

20. $\dfrac{-8\cdot 1\cdot 4\cdots(3r-5)}{12^r\cdot r!}\cdot x^r.$

21. $(r+1)x^{2r}.$

22. $\dfrac{r+1}{2^{r+2}}(3k)^r.$

23. $(-1)^r\dfrac{1\cdot 4\cdot 7\cdots(3r-2)}{3^{2r+1}\cdot r!}\cdot a^r.$

24. $(-1)^r\dfrac{(r+1)(r+2)(r+3)(r+4)}{24}\cdot x^{2r}.$

25. $1/2-x^2/2+x^4-7x^6/3+\cdots$.

26. $\dfrac{2\cdot 5\cdot 8\cdots(3r-1)}{9(r!)}\cdot a^r$; $1/9+2a/9+5a^2/9+\cdots$.

28. $t_{20}=210x^{19}$; $t_{62}=1953x^{61}$;$171x^{17}$; $5151x^{100}$;
$\frac{1}{2}n(n+1)x^{n-1}$; $\frac{1}{2}(3m+2)(3m+3)x^{3m+1}$.

29. $1+2x+2x^2+2x^3+\cdots$; $|x|<1$; $2x^r$.

30. $x-x^2-3x^3+\cdots$.

31. $x-5x^2+9x^3+\cdots$.

32. $4+12x+24x^2+\cdots$.

33. $4-12x+24x^2+\cdots$.

34. $-197x^{100}$; $(3-2n)x^n$.

35. $-576x^{50}$; $\frac{1}{2}(-1)^{r-1}(r^2-27r+2)x^r$.

36. $1+x-7x^2-8x^3+\cdots$; for all values of x.

37. $5003x^{50}$; $(2n^2+3)x^n$.

38. $6x^{50}$; $6x^n$; $n>1$.

39. 1.00995; 0.9151.

40. $17-9x+5x^2-\frac{20}{3}x^3+\cdots$.

41. (*a*) $3+3x+9x^2+\cdots+\{2^{n+1}+(-1)^n\}x^n+\cdots$,
true for $|x|<\frac{1}{2}$;
(*b*) $1+4x-\frac{17}{2}x^2+\cdots+\{2^{1-n}-(-3)^n\}x^n+\cdots$,
true for $|x|<\frac{1}{3}$.

42. $\frac{(r+1)(r+2)(r+3)}{3!}x^r$; $1+4x+10x^2+20x^3+\cdots$.

43. If p is a multiple of 4.

44. $1-18k+135k^2-540k^3+1215k^4-1458k^5+729k^6$;
$1+3p+6p^2+7p^3+6p^4+3p^5+p^6$.

45. 3; $3(-1)^5$.

46. $x^{-1}+x^{-2}+x^{-3}+x^{-4}+\cdots$, true for $|x|>1$.

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1. (*a*) 5.0990; (*b*) 9.99667; (*c*) 2.0033; (*d*) 121.2;
(*e*) .20006; (*f*) 49.3261.

2. (*a*) $4-\frac{37}{3}x$; (*b*) $10-\frac{209}{4}x$.

Exercises on Chapter IX—Page 271

1. $1+x/2-x^2/8+x^3/16+\cdots$; $|x|<1$.

2. $x^5-10x^4a+40x^3a^2-80x^2a^3+80xa^4-32a^5$.

3. $1-x^3+x^6-x^9+x^{12}+\cdots$; $|x|<1$.

4. $1-5x+5x^2/2+5x^3/2+\cdots$.

5. $2+4x/3-8x^2/9+80x^3/81+\cdots$; $|x|<\frac{1}{2}$.

6. $x^{20}-60x^{21}+1710x^{22}-30{,}780x^{23}+\cdots$.

7. $8-12x+12x^2-10x^3+15x^4/2+\cdots$; $|x|<2$.

8. $1/8+9x/32+135x^2/256+\cdots$.

9. $(-1)^r {}_7C_r(2q)^r \cdot 3^{7-r}$; 15,120$q^4$.

10. $(-1)^r \dfrac{(r+1)(r+2)(r+3)}{6} \cdot 3^r x^r$, $2835x^4$.

11. $\dfrac{-2 \cdot 1 \cdot 4 \cdots (3r-5)}{r!} \cdot 4x^r$, $-28x^4/3$.

12. 693/256. **13.** 5.03968.

15. (*a*) $1+2x$; (*b*) $32+160x$.

16. t_6; no. **17.** 23.

18. $x+4x^2+10x^3+ \cdots$; $(6r-8)x^r$.

19. $1077x^{50}$; $\frac{1}{2}(-1)^n(n^2-7n+4)x^n$.

20. $8+67x/3$.

21. (*a*) t_{31}; (*b*) t_{21}; (*c*) no term.

22. ${}_{20}C_{q-1} \cdot s^{2q-22}$. **23.** 1.0000995.

24. $729e^3-486e^{11/2}+135e^8-20e^{21/2}+ \cdots$.

25. $2x+8x^2+26x^3+80x^4+ \cdots$; 3^m-1;
expansion true for $-\frac{1}{3}<x<\frac{1}{3}$.

26. 3.98. **27.** $84k^9x^{12}$.

28. $z-\dfrac{z^2}{4}+\dfrac{z^3}{4}+ \cdots$; -4^{-k}; 4^{1-k}.

29. .0000246. **30.** (*a*) 16; (*b*) 7776.

31. $1+\frac{2}{3}x+\frac{8}{9}x^2$; 1.00676. **32.** .9057; 2.991.

34. 803; $\frac{1}{3}(m+1)(2m^2+m+9)$.

37. 14/3; 15.

38. $\dfrac{1 \cdot 3 \cdot 5 \cdots (2r-1)}{r!} x^{2r}$; $1+x^2+\frac{3}{2}x^4+\frac{5}{2}x^6+ \cdots$,
true for $|x| < \sqrt{2}/2$.

42. (*a*) $x^{-2}+2x^{-3}+3x^{-4}+4x^{-5}+\ \cdot\ \cdot\ \cdot$, true for $|x| > 1$.
(*b*) $\frac{1}{27}x^{-3}-\frac{4}{27}x^{-4}+\frac{32}{81}x^{-5}-\frac{640}{729}x^{-6}+\cdots$,
true for $|x|>\frac{4}{3}$.

43. $1+5x+\frac{31}{2}x^2+\ \cdot\ \cdot\ \cdot$, true for $|x|<\frac{1}{4}$.

46. $(1+2x)^8$.

49. 2^n.

52. $\sqrt[3]{3/2}$.

54. (*c*) t_8; t_5 and t_6.

55. (*b*) (i) t_{18} and t_{19}; (ii) t_5.

56. 6336.

57. t_{10}.

60. $9x^{10}$; $(n-1)x^n$; expansion holds for $|x| < 1$.

62. $\frac{1}{9}\{1-(6n+7)2^{2n+2}\}$.

63. $\dfrac{(3m)!}{m!(2m)!}$.

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3. (*a*) \$6468.0, \$1468.0; (*b*) \$6476.3, \$1476.3.

4. (*a*) 3.023%; (*b*) 4.06%.

5. (*a*) £1534.8; (*b*) £1525.7; (*c*) £1521.0.

6. (*a*) 11.9 years; (*b*) 11.7 years.

7. \$2025.2.

8. \$3642.2.

9. \$195.02.

10. 14.

11. 31.9.

12. \$5273.0.

14. \$312.22.

15. £1698 4s.

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1. (*a*) \$8316.6; (*b*) 37,976 francs.

2. \$10,489.

3. £5543.7.

4. \$10,000.

5. \$13,874.

6. \$10,539.

7. \$133.82.

8. \$703.61.

10. The first.

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1. \$947.58.
3. \$522.90.
5. \$8110.6.
6. 107.15.
7. Yes.
8. \$1131.1.
9. \$186.21.
10. \$4545.4.
11. \$104.31.
12. 4.63%.

Page 304

1. \$4271.4.
2. \$5861.5.
4. \$1039.6.
5. \$441.39.
6. \$685.55; \$1084.5.
7. (*a*) \$13,802; (*b*) \$13,488.

Exercises on Chapter X—Page 305

1. (*a*) 37,006 francs; (*b*) 37,149 francs; (*c*) 37,222 francs.
2. (*a*) \$7894.1; (*b*) \$7880.3; (*c*) \$7873.3.
3. (*a*) 4.04%; (*b*) 5.095%.
4. £2541.1.
5. £4590.3.
6. \$952.72.
7. \$8846.0.
8. \$101.86.
9. (*a*) \$110.48; (*b*) \$160.48.
10. \$154.72.
11. \$421.35.
12. \$72.41.
13. \$2578.0.
14. \$67,574.
15. \$9606.8.
16. \$3983.6; \$3004.3.
17. \$3671.4.
18. \$116,500.
19. \$32.29; \$9.31.
20. Decreased by \$247.55.
21. 31 years, nearly.
22. \$596.20.

23. 3.2%. **24.** \$2130.

25. \$1048.5. **26.** 4.3%.

27. 5%. **28.** \$3649.4.

29. 2.5%.

30. (*a*) \$540.55; (*b*) \$562.37; 3.8%.

32. Sell; 2.9%.

33. 2.78% per month; 33.4% per annum.

Additional Exercises—Page 319

4. (*a*) $x>\frac{3}{2}$ and $-\frac{3}{2}<x<1$; (*b*) $x<-\frac{1}{2}$ and $\frac{1}{3}<x<3$.

6. $\frac{1}{6}n(n+1)(n+2)$.

8. $-2(x-1)(x-3)(x+4)(3x-2)$.

9. $\dfrac{m!(m+1)!}{(m-w+1)!}$. **10.** $n=10$; t_3, t_4, t_5.

12. $(-1\pm\sqrt{3})/2$.

13. (*a*) $27x^2-15px+125=0$; (*b*) $3x^2+px-2p^2+125=0$;
(*c*) $(2p^2+15)x^2-9px+9=0$.

14. $9n$.

17. (*a*) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

18. (*a*) 163; (*b*) $\frac{1}{2}(2^{2n}+{}_{2n}C_n)$. **20.** (*a*) 44.4%; (*b*) 35.1%.

22. $\frac{1}{6}n(2n^2+3n+13)+2^{n+1}-2$.

24. (*b*) $(x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)$.

26. $-\frac{1}{2}$, -2, $\frac{1}{2}(3\pm\sqrt{5})$. **29.** 25,200.

30. (*a*) $2+5x+9x^2+\cdots+(4r+1)x^r+\cdots$, true for $|x|<1$:
(*b*) $1+3x^{-1}+7x^{-2}+\cdots+(4r-1)x^{-r}+\cdots$,
true for $|x|>1$.

31. $2^{2^{n+1}}-1$.

32. 2, -6, 18.

35. $x^2-10x+35$.

37. $5'' \times 10''$.

38. $\frac{1}{8}n(n-1)(n-2)(n-3)$.

39. 756.

42. $P=xy+2x^2y^2$.

44. x^2-2x+4.

46. -3, 1, 5.

47. $\frac{1}{30}\,n(n+1)\,(6n^3+9n^2+n-1)$.

48. 1296.

49. $\dfrac{52!}{2!2!3!45!}$.

50. -8^{-n}; 2^{-3n-2}.

51. 1 and 5, or 1 and -6.

53. $x^2+y^2=1$.

55. (*a*) $\frac{1}{9}\,(10^{n+1}-9n-10)$; (*b*) $\frac{4}{81}\,(10^{n+1}-9n-10)$.

57. $3(x-2)^3+5(x-2)^2-2(x-2)+4$.

59. 30.

63. (*a*) $\frac{1}{2}m(m^2+1)$.

64. $|y| \leqq 2/\sqrt{3}$.

67. $(3x+5y-4z)(2x-y+3z)$.

68. 286.

69. $(1+x/3)^{-9}$.

70. 15 payments, last of which is $105.36.

71. $\dfrac{p}{6}\,\{2p^2-3p(m+n+1)+3(m+n)+6mn+1\}$.

74. (*a*) $b^2=4c$; (*b*) $b^2=3c$, $b^3=27d$.

75. $\dfrac{5}{2x-5}-\dfrac{7}{6x-2}$.

77. $(0, 0, 0)$, $(0, \pm 2\sqrt{3}, 0)$, $(\pm 2i\sqrt{3}, 0, 0)$, $(0, 0, \pm 2\sqrt{3})$, $(2, -4, -4)$, $(-2, 4, 4)$.

78. $\dfrac{(2n)!}{4\,(n^2-k^2)}$.

79. $-2(k^2-1)(x-y)(y-z)(z-x)$.

80. $\sqrt[3]{4}$. **83.** $b=\frac{1}{2}a^2-2$.

85. $\dfrac{1+r-(n+1)^2r^n+(2n^2+2n-1)r^{n+1}-n^2r^{n+2}}{(1-r)^3}$.

86. (11, 60, 61), (60, 11, 61). **88.** ${}_{3n}C_n\{3\cdot 2^{2n}-2\cdot {}_{2n}C_n\}$.

89. $a=d=1$, $b=c=11$, $e=0$. **90.** 3.5%.

91. 5, 4, 3; 13, 5, 12; 10, 8, 6; 25, 7, 24; 20, 12, 16; 17, 15, 8; 29, 21, 20; 26, 24, 10.

94. $(x-2)(x-4)(4x^2-5x+3)$.

95. (1, 2, 4), (1, 4, 2), (2, 4, 1), (2, 1, 4), (4, 1, 2), (4, 2, 1).

98. 1.75. **99.** $\dfrac{(m+n)!}{m!\,n!}$.

Review of Basic Skills—Page 339

Set No. 1—Page 339

1. 56. **2.** 16. **3.** -20. **4.** 224.

5. 45. **6.** 5. **7.** 0. **8.** 75.

9. -21. **10.** 1/2. **11.** $-1/3$. **12.** 0.

13. 1. **14.** -120.

15. (i) 3; (ii) $2-\frac{1}{2}\sqrt{14}$; (iii) -1.

16. (*a*) b^2-3b+7; (*b*) $c^2+d^2+2cd-3c-3d+7$; (*c*) $(1-3a+7a^2)/a^2$.

17. (*a*) $(x+4)/(x-4)$; (*b*) $(1-4x)/(1+4x)$; (*c*) $(x+y-4)/(x+y+4)$.

18. $2a^2+2b^2$.

19. (*a*) $-1/2$; (*b*) $y/(y^2+1)$; (*c*) $(y+1)/(y^2+2y+2)$.

20. (*a*) 0; (*b*) 0; (*c*) $2a+1$.

Set No. 2—Page 340

1. $7a-9b+12c$.
2. $8x-6y$.
3. $2x^2-7xy+9y^2$.
4. $9z^3-8z^2+8z-13$.
5. $3x^2+7ax$.
6. $2x^2+3xy+7y^2$.
7. $3b-12c$.
8. $2m^3$.

Set No. 3—Page 340

1. $-15x^3y^4$.
2. $18a^3b^4$.
3. $6x^2+19x-7$.
4. $63m^4-24m^2n-15n^2$.
5. a^3+8b^3.
6. $64x^2-9y^2$.
7. $a^2+4ab+4b^2-c^2$.
8. $9m^4-4n^4$.
9. a^3+b^3.
10. $z^4+2z^3+z^2-1$.
11. $16a^2+40ab+25b^2$.
12. $4x^2y^2-12xyc+9c^2$.
13. $a^3+3a^2b+3ab^2+b^3$.
14. $x^2+y^2+z^2+2xy+2yz+2zx$.
15. $x^3-3x^2y+3xy^2-y^3$.
16. $1-6x+12x^2-8x^3$.
17. $4x^2+9y^2+16z^2-12xy$ $-24yz+16xz$.
18. $27m^3+27m^2n+9mn^2+n^3$.
19. $9x^2y^4-12xy^2z+4z^2$.
20. $25m^2n^2+30+9/m^2n^2$.
21. a^2+2+1/a^2.
22. $x^3-3x+3/x-1/x^3$.
23. $-7s^2/2$.
24. $4a^2b$.
25. $-ab+b^2+5$.
26. $3x^2+4x+10$.
27. $3x^2-2x+1$.
28. x^2+xy+y^2.
29. $x^2-2xy-3y^2$.
30. $2a^2-a-2$.
31. $-7x^2-36xy+10y^2$.
32. $-2x^2+110x+380$.
33. $4km-8mn+4kn$.
34. $-41x^3+30x^2-51x+74$.

Set No. 4—Page 342

1. $(x-3)(x-2)$. 2. $(2x-y)(2x+y)$.

3. $(2a+b)(4a^2-2ab+b^2)$. 4. $(x-m)(p-1)$.

7. $(m+5)(m+3)(m-3)$.

8. $2a(3a+4b)(9a^2-12ab+16b^2)$.

9. $x^2(x+2)(x-2)$. 10. $4x^2y^2z^2(z-2x+3)$.

11. $x(3x-4y)(9x^2+12xy+16y^2)$.

12. $(2x+5)(2x-3)$. 15. $(1+a-b)(1-a+b)$.

16. $(m+2)(m-2)(m+1)(m-1)$.

17. $(a+b)(a-b+1)$. 18. $2(a-13)(a+2)$.

19. $x(x+5)(m-n)$. 20. $(3x+4)(2x-1)$.

21. $(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$.

22. $(3x+4)(7x-6)$.

23. $(a+b)(a-b)(a^2+b^2)(a^2+ab+b^2)(a^2-ab+b^2)$
$(a^4-a^2b^2+b^4)$.

24. $(4a-3)(3a-8)$. 26. $(2x^2-x-1)(2x^2+x-1)$.

27. $(2m^2+2m+1)(2m^2-2m+1)$.

28. $(x-2y)(x^2-3xy+4y^2)$. 29. $3a(3b-7c)(3b+7c)$.

30. $(2y+3z)(3y-5z)$.

31. $(3x+2y-4z)(3x-2y+4z)$.

32. $3y^2(3x-5)(2x-3)$. 33. $(2p-3q)(x-5y)$.

34. $(4a+3)(3a+5)$.

35. $7a^4(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.

36. $(a^2+3ab-3b^2)(a^2-3ab-3b^2)$.

37. $(m^2+4mn-n^2)(m^2-4mn-n^2)$.

40. $(2a-3)(3a+2)(a+5)$.

Set No. 5—Page 344

1. $\dfrac{m}{m+x}$.
2. $\dfrac{x-5}{x-3}$.
4. $\dfrac{b-a}{m}$.
5. a^2+2a+4.
6. $\dfrac{4x(5x+3)}{3y(5x-3)}$.
7. $\dfrac{1}{9x^2+12x+16}$.
8. $\dfrac{x-y-1}{x+y}$.
9. $\dfrac{1}{x^2-y^2}$.
10. $\dfrac{x+y}{xy}$.
11. $\dfrac{x+y}{y-x}$.
12. $\dfrac{2y}{x^2-y^2}$.
13. $\dfrac{1}{x^2-y^2}$.
14. $\dfrac{-a^2+12a+2}{24a^2}$.
15. $\dfrac{a-b}{a+2b}$.
17. $\dfrac{m^2+4}{m^2-1}$.
18. $\dfrac{(x-y)^2}{x^2-xy+y^2}$.
19. $\dfrac{x^2+9x+14}{x-2}$.
20. $\dfrac{2x^2-7xy+6y^2}{7xy-6x^2-2y^2}$.
21. $\dfrac{15}{a^2-25}$.
22. $\dfrac{x+y}{x^2+2x+4}$.
23. $\dfrac{x}{6x+7}$.
24. $1/(m^2-1)$.
25. 0.
26. x/y.
27. x^2.

Set No. 6—Page 346

1. $\sqrt{8}$.
2. $\sqrt{27}$.
3. $\sqrt{20}$.
4. $\sqrt{108}$.
5. $\sqrt{150}$.
6. $\sqrt{18}$.
8. $2\sqrt{2}$.
9. $5\sqrt{3}$.
10. $5\sqrt{5}$.
11. $a\sqrt{b}$.
12. $2\sqrt[3]{2}$.
13. $6\sqrt{3}$.
14. $3\sqrt[3]{3}$.
15. $xy\sqrt{xy}$.
16. $10\sqrt{3}$.
17. $21\sqrt{6}$.
18. $-14\sqrt{5}$.
19. $2\sqrt{7}-8\sqrt{10}$.
20. $9\sqrt[3]{2}$.
21. $4\sqrt[4]{5}$.
22. $3\sqrt{2}$.
23 $6\sqrt{3}$.
24. 0.
25. $-2\sqrt[3]{3}$.

26. $26\sqrt{2}$. 27. $5\sqrt{5}$. 29. $13-\sqrt{15}$.

30. -9. 31. 2. 32. $8-16\sqrt{6}$.

33. -46. 34. $32\sqrt{15}$. 35. $32-24\sqrt{5}$.

36. $31-12\sqrt{6}+6\sqrt{2}-4\sqrt{3}$. 39. $\sqrt{2}/2$.

40. $\sqrt{3}$. 41. $-2(\sqrt{3}+\sqrt{2})$. 42. $7(3+2\sqrt{2})$.

43. $3(6+\sqrt{35})$. 44. $9-2\sqrt{14}$.

45. $-(37+6\sqrt{30})/17$. 46. $\frac{1}{2}(9-\sqrt{33})$.

Set No. 7—Page 348

1. -9. 2. $b-a$. 3. 3.

4. $a-b$. 5. $6/7$. 6. b/a.

7. -12. 8. ab. 9. $\frac{2}{3}\sqrt{3}$.

10. 4. 11. $c/(a-b)$. 12. $-5(\sqrt{3}+\sqrt{2})$.

13. $(m-n)/k$. 14. 0.5. 15. 20.

16. $-7/4$. 17. 4. 18. $3/2$.

19. -3. 20. $6/7$. 21. -5.

23. 4. 24. 4. 25. $2, 3$.

26, $2, 3$. 27. $8, 2$. 28. $0.3, -0.2$.

29. $1/2, 1/5$. 30. $-2, 3$. 31. $2, 3, 4$.

32. $5, 0, 1$. 33. $-3, 1, 2$. 34. $1/2, 1/3, 1/4$

35. $x>4$. 36. $x>-4$. 37. $x<-2$.

38. $x>-3$. 39. $x>7/4$. 40. $x<-2$.

41. $x<2$. 42. $x>-3$. 43. $-2>x>2$.

44. $-5\geqq x\geqq 5$. 45. $x<4$. 46. $x\geqq -10$.

47. $-7\leqq x\leqq 7$. 48. $-3\leqq x\leqq 3$.

Set No. 8—Page 350

1. $1/x^5$.
2. $\sqrt{a}$.
3. $\sqrt[4]{b^3}$.
4. $1/\sqrt[4]{m}$.
5. 1.
6. $1/\sqrt[5]{n^6}$.
7. $1/r^{100}$.
8. $1/e^{at}$.
9. $1/\sqrt[3]{x^2}$.
10. $1/k$.
11. $\sqrt[3]{p}$.
12. $\sqrt{y^5}$.
13. $5^{1/2}$.
14. $10^{1/3}$.
15. $2.4^{1/5}$.
16. $6^{-1/2}$.
18. $5^{-3/2}$.
19. $6^{3/5}$.
20. $a^{3/2}$.
21. 2.
22. 2.
24. 1/2.
25. 25.
26. 9.
27. 32.
28. 1/27.
29. $10\frac{7}{20}$.
30. 19.
31. $4x^2+3x-x^{-1}$.
32. $x+y$.
33. $6x^{2/3}+x^{1/3}-6-7x^{-1/3}-6x^{-2/3}+6x^{-1}$.
34. $3a^{1/5}+a^{-1/5}-6a^{-3/5}$.
35. (*a*) $(1+3x)^{1/3}$; (*b*) $(1+x)^{3/2}$.
36. (*a*) $(1-x)^{-1/2}$; (*b*) $(1+2x)^{-2/3}$; (*c*) $(1+x)^{-3}$.
37. $2x+2+x^{-1}$.
38. $5x^{2/3}-3+4x^{-2/3}$.
39. $-x+y+4x^{1/2}y^{1/2}$.
40. $-2a+8a^{1/3}$.
41. $x^{1/4}-1+x^{-1/4}$.
42. $x^{2/3}+2x^{1/2}+1+2x^{-1/2}+x^{-2/3}$.
43. 23.
44. 2^{n^2}.

Set No. 9—Page 351

26. $(-1, 3)$.
27. $(5\frac{2}{5}, -\frac{2}{3})$.
28. $(4, 3)$.
29. $(4, 3)$, $(3, 4)$.
30. $(0, -4)$, $(2, 0)$.
31. $(1, 3)\,(-1, -3)$.

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